

# Dependence in Probability, Analysis and Number Theory: The Mathematical Work of Walter Philipp (1936 - 2006)

ISTVÁN BERKES and HEROLD DEHLING

## Introduction

On July 19, 2006, Professor Walter Philipp passed away during a hike in the Austrian Alps as a result of a sudden heart attack. By the time of his death, Walter Philipp had been for almost 40 years on the faculty of the University of Illinois at Urbana-Champaign, for the last couple of years as professor emeritus. He is survived by his wife Ariane and his four children, Petra, Robert, Anthony and André. Walter Philipp is sorely missed by his family, but also by his many colleagues, coauthors and former students all over the world, to whom he was a loyal and caring friend for a long time, in some cases for several decades.

Walter Philipp was born on December 14, 1936 in Vienna, Austria, where he grew up and lived for most of the first 30 years of his life. He studied mathematics and physics at the University of Vienna, where he obtained his Ph.D. in 1960 and his habilitation in 1967, both in mathematics. From 1961 until 1967 he was scientific assistant at the University of Vienna. During this period, Walter Philipp spent two years as a postdoc in the US, at the University of Montana in Missoula and at the University of Illinois. In the fall of 1967 he joined the faculty of the University of Illinois at Urbana-Champaign, where he would stay for the rest of his life. Initially, Walter Philipp was on the faculty of the Mathematics Department, but in 1984 he joined the newly created Department of Statistics at the University of Illinois. From 1990 until 1995 he was chairman of this Department. While on sabbatical leave from the University of Illinois, Walter spent longer periods at the University of North Carolina at Chapel Hill, at MIT, at Tufts University, at the University of Göttingen and at Imperial College, London.

Walter Philipp received numerous recognitions for his work. Most outstanding of these was his election to membership in the Austrian Academy of Sciences.

As a student and postdoc at the University of Vienna, Walter Philipp worked under the guidance of Professor Edmund Hlawka, founder of the famous postwar Austrian school of analysis and number theory. Other former students of Professor Hlawka include Johann Cigler, Harald Niederreiter, Wolfgang M. Schmidt, Fritz Schweiger and Robert Tichy. It was here that Walter Philipp got in touch with the classical topics from analysis and number theory that would guide a large part of his research for the rest of his life. Uniform distribution, discrepancy of sequences, number-theoretic transformations associated with various expansions of real numbers, additive number-theoretic functions, Diophantine approximation, lacunary series, became recurrent themes in Walter Philipp's subsequent work. He studied these themes using techniques from probability theory, e.g. for mixing processes, martingales and empirical processes. He contributed greatly to the development of several branches of probability theory and solved much investigated, difficult problems in analysis and number theory with the help of the tools he developed.

## Themes from Analysis and Number Theory

A full understanding and appreciation of Walter Philipp's research requires the background of some topics from analysis and number theory. In what follows we shall briefly introduce the

topics that were recurrent themes in Walter Philipp's work.

**UNIFORM DISTRIBUTION MOD 1.** A sequence  $(x_n)_{n \geq 1}$  of real numbers is called uniformly distributed mod 1 if for all  $x \in [0, 1]$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \leq x\} = x,$$

where  $\{x\}$  denotes the fractional part of  $x$ . More generally, a sequence of vectors  $(x_n) \subset \mathbb{R}^d$  is called uniformly distributed mod 1 if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \in A\} = \lambda(A)$$

for all rectangles  $A \subset [0, 1]^d$ , where  $\lambda$  denotes Lebesgue measure and for  $x \in \mathbb{R}^d$  the symbol  $\{x\}$  is interpreted coordinatewise. The famous Weyl criterion (1916) states that a necessary and sufficient condition for the uniform distribution of  $(x_n)$  modulo 1 is

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i \langle k, x_n \rangle} = 0$$

for all  $k \in \mathbb{Z}^d \setminus \{0\}$ . An immediate consequence is the uniform distribution of the sequence  $(n\alpha)_{n \geq 1} \subset \mathbb{R}$  for irrational  $\alpha$ , but with suitable methods this leads to the uniform distribution of many other sequences in one and higher dimension, for example, a large class of sequences of the type  $\{n_k \alpha\}$  for increasing sequences  $(n_k)$  of positive integers.

**DISCREPANCIES.** Given a sequence of real numbers  $(x_n)_{n \geq 1}$  uniformly distributed mod 1, one can study the discrepancy

$$D_N := \sup_{x \in [0, 1]} \left| \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \leq x\} - x \right|.$$

A Glivenko-Cantelli type argument shows that  $D_N \rightarrow 0$ , and one may then ask for the exact rate of convergence of  $D_N$  to 0. In the case of a  $d$ -dimensional sequence  $(x_n)_{n \geq 1}$ , one can define the discrepancy with respect to a class  $\mathcal{C}$  of subsets of  $[0, 1]^d$  by

$$D_N(\mathcal{C}) := \sup_{A \in \mathcal{C}} \left| \frac{1}{N} \#\{1 \leq i \leq N : \{x_i\} \in A\} - \lambda(A) \right|.$$

In this case, the additional issue of the choice of suitable class  $\mathcal{C}$  arises. Note that  $D_N(\mathcal{C}) \rightarrow 0$  does not necessarily hold even if the sequence  $(x_n)_{n \geq 1}$  is uniformly distributed.

**$\theta$ -ADIC EXPANSION OF REAL NUMBERS.** Let  $\theta > 1$ , not necessarily an integer. Every real number  $\omega \in [0, 1)$  can be written as an infinite series

$$\omega = \sum_{n=1}^{\infty} \theta^{-n} x_n$$

where  $0 \leq x_n < \theta$  are integers. Clearly  $x_n = [T^n \omega]$  where the transformation  $T : [0, 1] \rightarrow [0, 1]$  is defined by  $T\omega := \{\theta \omega\}$  and  $[x]$  denotes the integer part of  $x$ . Also,  $T^n \omega = \sum_{k=1}^{\infty} \theta^{-k} x_{n+k}$ .

The basic asymptotic question here is the distribution of digits in the expansion, for example, one can ask if the limits of relative frequencies

$$F_k := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1_{\{x_n=k\}}$$

exist. If  $\theta$  is an integer, the transformation  $T$  is ergodic and has Lebesgue measure as an invariant measure, thus the ergodic theorem implies that the limits  $F_k$  exist and are equal to  $\theta^{-1}$  for  $k = 0, 1, \dots, \theta - 1$  and almost every real number  $\omega$ . With the usual terminology, almost every real number is normal with respect to base  $\theta$ . This statement, proved first by Borel in 1909, is a typical result in the metric theory of numbers, stating that a certain property holds for almost every real number, without specifying the exceptional set. In fact, to determine whether a given number  $\omega$  is normal is a very hard problem and only very few normal numbers are known explicitly. We do not know, for example, if  $\sqrt{2}$ ,  $e$  or  $\pi$  are normal in any base. Note that the normality of  $\omega$  in base  $a \in \mathbb{N}$  is equivalent to the statement that the sequence  $(a^n \omega)_{n \geq 1}$  is uniformly distributed mod 1.

If  $\theta$  is not an integer, the transformation  $T$  is still ergodic, but Lebesgue measure is not an invariant measure. It is known from work of Alfréd Rényi (1957) that there exists a unique invariant measure  $\mu$  which is equivalent to Lebesgue measure. In this case, the sequence  $(\theta^n \omega)_{n \geq 1}$  is no longer uniformly distributed, but

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : T^n \omega \leq x\} = \mu([0, x]),$$

for almost every  $\omega$ .

CONTINUED FRACTION EXPANSION. Every real number  $\omega \in (0, 1]$  can be expressed as an infinite continued fraction

$$\omega = \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \dots}}} = [x_1, x_2, \dots],$$

where  $x_i \in \{1, 2, \dots\}$ . Closely related is the transformation  $T : (0, 1] \rightarrow (0, 1]$ , defined by

$$T\omega := \{1/\omega\}.$$

The  $n$ -th digit in the continued fraction expansion is given by  $x_n = [T^n \omega]$ . As in the case of the  $\theta$ -adic expansion,  $T^n \omega$  can be written as a function of  $x_{n+1}, x_{n+2}, \dots$  by

$$T^n \omega = [x_{n+1}, x_{n+2}, \dots].$$

$T$  is an ergodic transformation with invariant measure given by the Gauss measure

$$\mu((a, b]) = \frac{1}{\log 2} \int_a^b \frac{1}{1+x} dx.$$

As a consequence, the asymptotic distribution of the sequence  $(T^n \omega)_{n \geq 1}$  is governed by the Gauss measure, i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : T^n \omega \leq x\} = \frac{1}{\log 2} \int_0^x \frac{1}{1+t} dt.$$

From here, one obtains that the integer  $k$  occurs in the continued fraction expansion of a random number  $\omega$  with relative frequency  $\frac{1}{\log 2} \left( \log \frac{k}{k+1} - \log \frac{k+1}{k+2} \right)$ , a fact already conjectured by Gauss.

ADDITIVE FUNCTIONS IN NUMBER THEORY. A function  $f : \mathbb{N} \rightarrow \mathbb{R}$  is called additive if

$$f(mn) = f(m) + f(n),$$

whenever  $m$  and  $n$  are coprimes. A simple example of an additive function is  $\omega(n)$ , the number of prime divisors of  $n$ . Hardy and Ramanujan studied this function and showed that  $\omega(n)$  is of the order  $\log \log n$ . More precisely, if  $P_N$  denotes the uniform distribution on the first  $N$  integers, then

$$P_N \left( n \leq N : \left| \frac{\omega(n)}{\log \log N} - 1 \right| \geq \epsilon \right) \rightarrow 0$$

for any  $\epsilon > 0$ . Turán observed that the Hardy-Ramanujan theorem is a simple consequence of an easily verifiable inequality for the second moment of  $\omega(n)$  and Chebychev's inequality and thus initiated the subject of probabilistic number theory. The Hardy-Ramanujan theorem was later strengthened by Erdős and Kac, who proved a central limit theorem

$$P_N \left( n \leq N : \frac{\omega(n) - \log \log N}{\sqrt{\log \log N}} \leq x \right) \rightarrow \int_{-\infty}^x e^{-y^2/2} dy.$$

DIOPHANTINE APPROXIMATION. Let  $f$  be a positive, continuous, nonincreasing function on  $\mathbb{R}^+$ . By a classical result of Khinchin, for almost all real  $\alpha$  the inequality

$$|q\alpha - p| \leq \frac{f(q)}{q} \tag{1}$$

has infinitely many or only finitely many solutions in integers  $p, q$  according as  $\sum \frac{f(k)}{k}$  diverges or converges. Probabilistically, this is the Borel-Cantelli lemma for certain dependent events; the main difficulty is to deal with the dependence in the case  $\sum \frac{f(k)}{k} = \infty$ . Khinchin's result has been generalized and improved upon in many directions by Cassels, W. M. Schmidt, Erdős, LeVeque, Szűsz, Gallagher, Ennola, Billingsley and many others. The simplest proof depends on a connection of the problem with continued fraction theory. Call a fraction  $p/q$  a best approximation to  $\alpha$  if it minimizes  $|q'\alpha - p'|$  over fractions  $p'/q'$  with denominator  $q'$  not exceeding  $q$ . The successive best approximations to  $\alpha$  are the convergents  $p_n(\alpha)/q_n(\alpha)$ ,  $n=1, 2, \dots$  of its continued fraction expansion and thus the value of  $|q\alpha - p|$  for the  $n$ -th in the series of best approximation is  $d_n(\alpha) = |q_n(\alpha)\alpha - p_n(\alpha)|$ . Thus the study of number of solutions of (1) is equivalent to the study of growth of the sequence  $d_n(\alpha)$ . Khinchin proved that

$$\frac{1}{n} \log d_n(\alpha) \rightarrow -\frac{\pi^2}{\log 2}$$

for almost every  $\alpha$ .

LACUNARY SEQUENCES. Probabilistic methods play an important role in harmonic analysis and there is a profound connection between probability theory and trigonometric series. From a purely probabilistic point of view, the trigonometric system  $(\cos 2\pi n x, \sin 2\pi n x)_{n \geq 1}$  is a sequence of orthogonal (i.e. uncorrelated) random variables over  $[0, 1]$ , which, however, are strongly dependent. For example, the r.v.'s  $\sin 2\pi n x$  have the same distribution, but their partial sums  $\sum_{n < N} \sin 2\pi n x$  remain bounded for any fixed  $x$ , a behavior very different from that of i.i.d. random variables. However, it has been known for a long time that for rapidly increasing  $(n_k)$ , the sequences  $(\sin 2\pi n_k x)_{n \geq 1}$  and  $(\cos 2\pi n_k x)_{n \geq 1}$  behave like sequences of independent random variables. For example, Salem and Zygmund (1947) proved that if  $(n_k)$  satisfies the Hadamard

gap condition  $n_{k+1}/n_k \geq q > 1$  ( $k = 1, 2, \dots$ ), then  $(\sin 2\pi n_k x)_{n \geq 1}$  obeys the central limit theorem, i.e.

$$\lim_{N \rightarrow \infty} \lambda\{x \in (0, 1) : \sum_{k \leq N} \sin 2\pi n_k x < t\sqrt{N/2}\} = (2\pi)^{-1/2} \int_{-\infty}^t e^{-u^2/2} du. \quad (2)$$

Erdős (1962) proved that the CLT (2) remains valid if the Hadamard gap condition is weakened to  $n_{k+1}/n_k \geq 1 + c_k/\sqrt{k}$ ,  $c_k \rightarrow \infty$  and this result is sharp. Similar results hold if  $\sin n_k x$  is replaced by  $f(n_k x)$ , where  $f$  is a real measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0, \quad \int_0^1 f^2(x) dx < \infty.$$

For example, if  $f \in \text{Lip}(\alpha)$ ,  $\alpha > 0$  and  $n_{k+1}/n_k \rightarrow \infty$ , then the CLT and LIL hold for  $f(n_k x)$ . (Takahashi (1961, 1963)). If we assume only  $n_{k+1}/n_k \geq q > 1$ , both the CLT and LIL can fail, a fact discovered by Erdős and Fortet. By a result of Gaposhkin (1970), given a sequence  $(n_k)$  satisfying the Hadamard gap condition,  $f(n_k x)$  satisfies the CLT and LIL for all smooth periodic functions  $f$  if and only if for any fixed nonzero integers  $a, b$ , the number of solution of the Diophantine equation  $\nu = an_k \pm bn_l$  ( $k, l \geq 1$ ) is  $O(1)$  uniformly in  $\nu > 0$ .

## Walter Philipp's work

Given that Walter Philipp published close to 80 research papers, it is impossible to mention every single result he ever obtained. We will instead try to focus on the main lines of his research. Philipp's earliest work, originating from his Ph.D. thesis, concerns uniform distribution mod 1. Weyl (1916) had shown that the sequence  $(a_n \omega)_{n \geq 1}$  is uniformly distributed modulo 1 for almost all  $\omega \in [0, 1]$ , if  $(a_n)_{n \geq 1} \subset \mathbb{R}$  is a sequence of positive numbers satisfying  $a_{n+1} - a_n \geq \delta > 0$  ( $n = 1, 2, \dots$ ) for some  $\delta > 0$ . Walter Philipp studied this question for  $d$ -dimensional sequences, i.e. for the sequence  $(A_n \omega)_{n \geq 1}$  where  $\omega \in \mathbb{R}^d$  and  $(A_n)_{n \geq 1}$  is a sequence of  $d$ -dimensional matrices satisfying some growth condition. As a corollary, uniform distribution of the sequence  $(A^n \omega)_{n \geq 1}$  for almost all  $\omega \in \mathbb{R}^d$  can be obtained, provided the matrix has all eigenvalues strictly larger than 1.

In 1967 Walter Philipp published the first in a series of papers on the asymptotic behavior of weakly dependent stochastic processes. This topic would dominate his research interests for the next 15 years and keep his close attention for the rest of his life. In the second half of the 1960s and during all of the 1970s Walter Philipp was internationally recognized as a leader in the development of new limit theory for weakly dependent processes and their applications to problems in analysis and number theory. In this period the focus of his research changed substantially: instead of an analyst and number theorist using tools from probability theory he became a probabilist applying his results to problems in analysis and number theory.

In all of the topics from analysis and number theory mentioned above, there are sequences of random variables in the background, most of them defined on the probability space  $[0, 1]$ , equipped with some measure equivalent to Lebesgue measure. In Weyl's almost sure equidistribution theory, we have the random variables  $\omega \mapsto a_n \omega$ . In each of the different expansions of real numbers  $\omega \in (0, 1]$ , the  $n$ -th digit maps  $\omega \mapsto x_n = x_n(\omega)$  are random variables, and so are the  $n$ -th iterates  $\omega \mapsto T^n \omega$ . With the notable exception of the digits in the expansion to an integer base  $a$ , none of these random variables are independent. But the dependence is weak, in some sense yet to be defined. Walter Philipp soon realized that the theory of weakly dependent

stochastic processes, then only recently created by publications of Rosenblatt (1956), Ibragimov (1962) and Billingsley (1968), provides the right framework for the problems he wanted to attack.

There is no such thing as a universal definition of weak dependence that would imply the validity of all limit theorems known for independent processes. There are many notions, each of them allowing, under additional technical assumptions, the proof of some of the classical limit theorems of probability theory. The stronger the notion, the more limit theorems can be established, but at the same time fewer examples satisfy the conditions. The earliest and classical notions of weak dependence are  $\alpha$ -mixing (also called strong mixing, but not to be confused with the same notion in ergodic theory) and  $\phi$ -mixing (also called uniform mixing). Let  $(X_n)_{n \geq 1}$  be a stochastic process, and define for integers  $k, l$  with  $k \leq l$  the  $\sigma$ -fields  $\mathcal{F}_k^l = \sigma(X_k, \dots, X_l)$ . We then define the mixing coefficients

$$\begin{aligned}\alpha(k) &:= \sup_{n \geq 1} \sup_{A \in \mathcal{F}_1^n, B \in \mathcal{F}_{n+k}^\infty} |P(A \cap B) - P(A)P(B)|, \\ \phi(k) &:= \sup_{n \geq 1} \sup_{A \in \mathcal{F}_1^n, B \in \mathcal{F}_{n+k}^\infty, P(A) > 0} \frac{|P(A \cap B) - P(A)P(B)|}{P(A)}.\end{aligned}$$

The process  $(X_n)_{n \geq 1}$  is called  $\alpha$ -mixing if  $\lim_{n \rightarrow \infty} \alpha(n) = 0$  and  $\phi$ -mixing if  $\lim_{n \rightarrow \infty} \phi(n) = 0$ . Rosenblatt (1956) and Ibragimov (1962) established central limit theorems for  $\alpha$ - and  $\phi$ -mixing random variables, requiring a combination of moment conditions and conditions on the speed at which the mixing rates converge to zero. Later, several other related mixing concepts ( $\psi$ ,  $\rho$  mixing, absolute regularity, etc.) were introduced and studied in detail. For stationary  $\phi$ -mixing processes  $(X_n)$ , Ibragimov conjectured that the central limit theorem holds if  $EX_1^2 < \infty$  and  $\text{Var}(\sum_{k=1}^n X_k) \rightarrow \infty$ , but until today this conjecture has not been verified. This conjecture inspired some of Walter Philipp's deepest results in the field of mixing random variables: his 1984 joint paper with Dehling and Denker, giving a necessary and sufficient condition for the CLT for  $\rho$ -mixing sequences without any rate or moment conditions and his 1998 joint paper with Berkes, giving a complete characterization of the law of the iterated logarithm and the domain of partial attraction of the Gaussian law for  $\phi$ -mixing sequences, again without any moment or rate conditions.

In his first papers on limit theorems for weakly dependent processes, culminating his 1975 AMS memoir with William Stout, Philipp solved the central limit problem (characterizing the limit distributions of arrays with corresponding criteria for convergence to specific limits) in the case of bounded variances, proved Berry-Esseen bounds for the speed of convergence in the CLT and obtained laws of the iterated logarithm under various weak dependence conditions. In addition to the mixing conditions already mentioned, he studied the  $\psi$ -mixing coefficients defined by

$$\psi(k) := \sup_{n \geq 1} \sup_{A \in \mathcal{F}_1^n, B \in \mathcal{F}_{k+n}^\infty} \frac{|P(A \cap B) - P(A)P(B)|}{P(A)P(B)},$$

and  $\psi$ -mixing processes. The notion of  $\psi$ -mixing is stronger than any of the other mixing conditions and  $\psi$ -mixing processes satisfy most of the classical i.i.d. limit theorems. The digits of a random number  $\omega \in (0, 1]$  in the continued fraction expansion form a  $\psi$ -mixing process. A second weak dependence condition investigated by Walter Philipp is a correlation condition for mixed products, requiring that for all integers  $0 \leq i_1 < \dots < i_r$ ,  $1 \leq j \leq r$ ,  $p_\nu \geq 0$

$$\left| E(X_{i_1}^{p_1} \cdot \dots \cdot X_{i_r}^{p_r}) - E(X_{i_1}^{p_1} \cdot \dots \cdot X_{i_j}^{p_j}) E(X_{i_{j+1}}^{p_{j+1}} \cdot \dots \cdot X_{i_r}^{p_r}) \right| \leq L(j)c(i_{j+1} - i_j) \sup_{1 \leq i \leq i_r} E|X_i|^{\sum p_\nu}.$$

The standard method to prove limit theorems for mixing processes, employed in the pioneering works of Rosenblatt (1956) and Ibragimov (1962), was the Bernstein blocking technique, giving an approximation of the characteristic function of partial sums of mixing sequences by the characteristic function of independent random variables via suitable correlation inequalities. This method leads to sharp results in the case of the CLT and LIL, but its applicability beyond them is rather limited: for example, upper-lower class refinements of the LIL require delicate tail estimates for the considered r.v.'s which are beyond the scope of the method. In his 1975 AMS Memoir with William Stout, Walter Philipp showed that sufficiently separated block sums of weakly dependent sequences are, after suitable centering, close to a martingale difference sequence, and thus using Skorohod embedding and Strassen's strong approximation technique, the partial sums of such sequences can be closely approximated by a Wiener process. This observation not only opens the way to prove a vast class of refined asymptotic results for mixing sequences, but the near martingale property can be easily verified for several other types of weak dependent processes for which the previous theory does not work, or leads to great difficulties: Markov processes, retarded mixing sequences, Gaussian processes, lacunary series, etc. In his 1979 joint paper with Berkes, Philipp made a further important step in the study of weak dependent behavior, showing that block sums of weakly dependent processes can be directly approximated by independent random variables, via the Strassen-Dudley existence theorem. This observation frees the investigations from moment conditions and works not only for real valued random variables, but for random variables taking values in abstract spaces. In the context of Banach space valued random variables, this method yields new results even for i.i.d. random variables, as a 1980 joint paper of Philipp and Kuelbs shows. This paper was the first in a long series of papers of Walter Philipp dealing with limit theorems of independent and weakly dependent  $B$ -valued random variables and Hilbert space valued martingales. The infinite dimensional setup also opens the way to study uniform Glivenko-Cantelli type results and uniform limit theorems for random variables indexed by sets, a popular and much studied topic in the 1970's and 1980's. Walter Philipp's contribution in this field is very substantial; see e.g. his profound joint paper with Dudley (1982). In his last paper written in 2006, Walter Philipp returns again to this topic, showing that metric entropy can be used to provide deep information on pseudorandom behavior and in the theory of uniform distribution. This paper completes a long circle in Philipp's mathematical work and at the same time opens a new direction in the study of weakly dependent behavior.

The importance of Walter Philipp's contributions to the asymptotic theory for weakly dependent processes can only be appreciated in the light of the many applications to problems in analysis and number theory. Some early applications are given in two papers entitled *Some metrical theorems in number theory* which are entirely devoted to such applications. In these papers Walter Philipp investigated the distribution of the sequence  $(T^n\omega)_{n\geq 1}$  for the maps  $T : (0, 1] \rightarrow (0, 1]$  associated with the  $\theta$ -adic expansion and the continued fraction expansion. If  $(I_n)_{n\geq 1}$  is a sequence of intervals,  $I_n \subset (0, 1]$ , one can study the quantity

$$A(N, \omega) := \sum_{n=1}^N 1_{\{T^n\omega \in I_n\}}.$$

If  $\mu$  denotes the invariant measure associated with  $T$ , then the expected value of  $A(N, \omega)$  becomes

$$\phi(N) = \sum_{n=1}^N \mu(I_n).$$

If all the intervals are identical, i.e.  $I_n = I$ , we obtain from the ergodic theorem that  $A(N, \omega) =$

$\phi(N) + o(N)$ . Walter Philipp sharpened this result considerably by showing that for any  $\epsilon > 0$ ,

$$A(N, \omega) = \phi(N) + O(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N)), \quad (3)$$

for almost all  $\omega \in (0, 1]$ . While the accuracy of this approximation is limited by the second order method used in the proof, shortly thereafter Philipp went much further: he observed that the considered sequences  $(T^n \omega)_{n \geq 1}$  are  $\phi$ -mixing with exponential rate and thus using blocking techniques and applying some of his asymptotic theorems obtained earlier, one can prove a whole series of highly attracting limit theorems for the digits in various expansions and Diophantine approximation, which not only improve several earlier results in the literature, but actually provide the precise asymptotics in a number of important questions of metric number theory. Let us formulate a few such results here. Let  $x = [a_1(x), a_2(x), \dots]$  be the continued fraction expansion of  $x \in (0, 1)$  and let  $\varphi(n) \rightarrow \infty$  be a sequence of integers with  $\sum \frac{1}{\varphi(n)} = \infty$ . Denote by  $A(N, x)$  the number of integers  $n \leq N$  with  $a_n(x) \geq \varphi(n)$  and put

$$\phi(N) = \frac{1}{\log 2} \sum_{n \leq N} \log \left( 1 + \frac{1}{\varphi(n)} \right).$$

Then

$$\lambda \left\{ x : \frac{A(N, x) - \phi(N)}{\sqrt{\phi(N)}} < z \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp(-t^2/2) dt$$

and for almost all  $x$

$$\limsup_{N \rightarrow \infty} \frac{|A(N, x) - \phi(N)|}{\sqrt{2\phi(N) \log \log \phi(N)}} = 1.$$

Also, letting  $L_N(x) = \max_{1 \leq k \leq N} a_k(x)$ , Philipp proved

$$\liminf_{N \rightarrow \infty} \frac{\log \log N}{N} L_N(x) = \frac{1}{\log 2}$$

for almost all  $x$ , verifying an old conjecture of Erdős. Further, let  $f$  be a continuous, positive, nonincreasing function on  $\mathbb{R}^+$  such that

$$\phi(N) = 2 \sum_{k \leq N} \frac{f(k)}{k} \rightarrow \infty$$

and let  $N_{\alpha, f}(n)$  denote the number of solutions of (1) in integers  $q \leq n$  and  $p$ . Then under mild additional regularity conditions on  $f$ , Walter Philipp proved

$$\lambda \left\{ x : \frac{N_{\alpha, f}(n) - \phi(n)}{\sqrt{\phi(n)}} < z \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp(-t^2/2) dt$$

and for almost all  $\alpha$

$$\limsup_{N \rightarrow \infty} \frac{|N_{\alpha, f}(n) - \phi(n)|}{\sqrt{2\phi(n) \log \log \phi(n)}} = 1.$$

A further much studied connection between probability theory and number theory is the distribution of values of additive functions. Walter Philipp's contribution in his field can be found in his 1971 AMS Memoir written with the ambitious goal to unify probabilistic number theory and to deduce at least the most typical results as special cases of limit theorems for mixing random variables. Because of the very different type of weak dependence conditions in number



theory, there is little hope that all applications of probability to number theory could be put in a general framework. But at least for applications in Diophantine approximation, continued fraction and related expansions, discrepancies and the distribution of additive functions, Walter Philipp succeeded in this program to a remarkable degree. He also studied weak convergence of additive function paths to Brownian motion, extending earlier results of Billingsley.

The basic motivation for the introduction of mixing conditions was to understand the asymptotic properties of weakly dependent structures in stochastics and driven by its intrinsic needs, the theory made a tremendous progress starting from the 60's and by now it is a closed, complete and beautiful theory, giving a nearly complete answer for the basic asymptotic questions connected with mixing structures. For a comprehensive treatise of the theory see the recent monograph of R. Bradley (2007).

While questions on weak dependence kept Walter Philipp's attention in his whole career, this did not prevent him from making fundamental contributions in other areas of probability theory, e.g. in the classical theory of independent random variables. In a short paper with M. Lacey in 1990 he proved that if  $(X_n)$  is a sequence of i.i.d. random variables with mean 0 and variance 1 then letting  $S_n = \sum_{k=1}^n X_k$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{k=1}^N \frac{1}{k} I \left\{ \frac{S_k}{\sqrt{k}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

with probability 1 for all  $x \in \mathbb{R}$ . This remarkable 'pathwise' form of the central limit theorem was already stated (without proof and without specifying conditions) by Lévy in 1937 and was proved by Brosamler and Schatte in 1988 under the assumption of higher moments. Due to these papers, almost sure central limit theory became extremely popular overnight and has not lost its attraction until today. The paper of Philipp and Lacey not only yields the final, optimal form of this theorem, but the method they used became the basic method in this field.

In a series of three papers, published in the mid 1980s jointly with Dehling and Denker, Walter Philipp investigated the asymptotic behavior of degenerate  $U$ -statistics. Given a symmetric function  $h : \mathbb{R}^m \rightarrow \mathbb{R}$  and an i.i.d. process  $(X_n)_{n \geq 1}$ , the  $m$ -variate  $U$ -statistic with kernel  $h$  is defined as

$$U_n(h) = \sum_{1 \leq i_1 < \dots < i_m \leq n} h(X_{i_1}, \dots, X_{i_m}).$$

The kernel is called degenerate if  $E(h(X, x_2, \dots, x_m)) = 0$  for almost all  $x_2, \dots, x_m$ . Dehling, Denker and Philipp proved a strong approximation of  $U_n(h)$  by  $m$ -fold Wiener-Itô integrals

$$I_n(h) = \int \dots \int h(x_1, \dots, x_m) dW(x_1) \dots dW(x_m),$$

where  $W(t)$  is a mean-zero Gaussian process. The results of this research enabled Dehling in a subsequent paper to establish the functional law of the iterated logarithm for degenerate  $U$ -statistics and for multiple Wiener-Ito integrals. In the course of their work, Dehling, Denker and Philipp also investigated the empirical  $U$ -process, defined by

$$\sqrt{n} \frac{1}{\binom{n}{m}} (\#\{1 \leq i_1 < \dots < i_m \leq n : h(X_{i_1}, \dots, X_{i_m}) \leq t\} - P(h(X_1, \dots, X_m) \leq t))_{t \in \mathbb{R}},$$

and established an almost sure invariance principle for this process.

The investigations on the asymptotic behavior of degenerate  $U$ -statistics lead directly to questions concerning the asymptotic behavior of certain Hilbert space valued martingales. For

the  $U$ -statistic applications, a bounded law of the iterated logarithm was sufficient. In later work, carried out jointly with Monrad, Walter Philipp established a Skorohod embedding of Hilbert space valued martingales.

Another favorite topic of Walter Philipp's research was lacunary series: he investigated such series already in his early papers in the 1960's and in his very last papers in 2006 he returned once more to this topic. By a well known theorem of H. Weyl (1916) quoted above, given any sequence  $(n_k)$  of positive numbers with  $n_{k+1} - n_k \geq \delta > 0$  ( $k = 1, 2, \dots$ ), the sequence  $\{n_k \omega\}$  is uniformly distributed mod 1 for almost all  $\omega$ . In contrast to the simplicity of this result, proving sharp bounds for the discrepancy of  $\{n_k \omega\}$  is very difficult, and the only precise results known before 1970 were the results of Khinchin (1923) and Kesten's (1964) for the case  $n_k = k$ . Kesten's result states that the discrepancy  $D_N(\omega)$  of the sequence  $\{k\omega\}$  satisfies

$$\lim_{N \rightarrow \infty} \frac{\log N \log \log N}{N} D_N(\omega) = \frac{2}{\pi^2} \quad \text{in probability.}$$

In 1975 Walter Philipp proved that if  $(n_k)$  is a sequence of integers satisfying the Hadamard gap condition  $n_{k+1}/n_k \geq q > 1$ , then the discrepancy  $D_N(\omega)$  of the sequence  $\{n_k \omega\}$  satisfies the law of the iterated logarithm, i.e.

$$C_1 \leq \limsup_{N \rightarrow \infty} \sqrt{\frac{N}{\log \log N}} D_N(\omega) \leq C_2 \quad (4)$$

for almost all  $\omega$ , where  $C_1 > 0$  is an absolute constant and  $C_2 = C_2(q) = 166 + 664(q^{1/2} - 1)^{-1}$ . This remarkable result verified a long standing conjecture of Erdős and Gál, and showed that, as far as its discrepancies are concerned,  $\{n_k \omega\}$  behaves like a sequence of independent random variables. For the partial sums of  $\sin n_k x$  such phenomena have already been observed by Salem and Zygmund in 1947, but the discrepancy situation is much more delicate: as in a later paper Philipp (1994) showed, for a suitable sequence  $n_k$  the limsup in (4) is greater than  $C \log \log \frac{1}{q}$  with an absolute constant  $C$  and thus for  $q$  close to 1 the limsup can be as large as we wish. Very recently, Fukuyama (2008) succeeded in computing the limsup for the sequences  $n_k = \theta^k$ ,  $\theta > 1$ .

For sequences  $(n_k)$  growing slower than exponentially, the LIL (4) is generally false, and the behavior of the discrepancy of  $\{n_k \omega\}$  becomes very complicated. R. C. Baker (1981) proved that for any  $n_k$ ,  $N D_N(\omega) = O((\log N)^{3/2+\varepsilon})$  for almost all  $\omega$  and Philipp and Berkes (1994) showed that the constant  $3/2$  here cannot be replaced by any number less than  $1/2$ . Despite these fairly precise results on the extremal behavior of discrepancies, the exact order of magnitude of  $D_N(\omega)$  for "concrete"  $n_k$  remains open. In one of his last papers, written jointly with Berkes and Tichy, Walter Philipp made a substantial step in clearing up this phenomenon as well: he showed that the asymptotic behavior of the discrepancy of  $\{n_k \omega\}$  is intimately connected with the number of solutions of Diophantine equations of the type  $a_1 n_{k_1} + \dots + a_p n_{k_p} = b$ . The discovery of this remarkable arithmetical connection is again a characteristic achievement of Walter Philipp, linking probabilistic phenomena with asymptotic results in analysis and number theory.

In conclusion we mention an interesting result of Philipp on the extremal behavior of exponential sums. From Carleson's convergence theorem for Fourier series in  $L^2$  it follows that if  $f$  is a nondecreasing positive function on  $\mathbb{R}^+$  satisfying

$$\sum_{k=1}^{\infty} \frac{1}{k f(k)^2} < \infty \quad (5)$$

then for any increasing sequence  $(n_k)$  of positive integers we have

$$\left| \sum_{k=1}^N e^{2\pi i n_k x} \right| = O(N^{1/2} f(N)) \quad \text{a.e.} \quad (6)$$

In particular, we have for any  $(n_k)$

$$\left| \sum_{k=1}^N e^{2\pi i n_k x} \right| = O(N^{1/2} (\log N)^{1/2+\varepsilon}) \quad \text{a.e.}$$

for any  $\varepsilon > 0$ . As early as 1930, Walfisz proved that for  $n_k = k^2$  the left hand side of (6) exceeds  $N^{1/2}(\log N)^{1/4}$  for infinitely many  $N$ , but this does not reach Carleson's upper bound. Walter Philipp proved that if  $f$  is a nondecreasing positive function on  $\mathbb{R}^+$  satisfying mild regularity conditions such that the sum in (5) diverges, then the exponential sum in (6) exceeds  $N^{1/2} f(N)$  a.e. for infinitely many  $N$ . This provides a complete solution of the problem of extremal speed of exponential sums and provides yet another example for the power of weak dependence techniques in problems of classical analysis.

## References

- [1] R. C. BAKER: Metric number theory and the large sieve. *Journal of the London Mathematical Society* **24**, 34–40 (1981)
- [2] R. C. BRADLEY: *Introduction to Strong Mixing Conditions*, Volume 1 – 3, Kendrick Press 2007.
- [3] G. BROSAMLER: An almost everywhere central limit theorem. *Mathematical Proceedings of the Cambridge Philosophical Society* **104**, 561–574 (1988)
- [4] P. BILLINGSLEY: *Convergence of Probability Measures* J. Wiley, New York 1968.
- [5] P. ERDŐS: On trigonometric sums with gaps, *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **7**, 37–42 (1962).
- [6] A. FISHER: Convex-invariant means and a pathwise central limit theorem. *Advances in Mathematics* **63**, 213–246 (1987).
- [7] K. FUKUYAMA: The law of the iterated logarithm for discrepancies of  $\{\theta^n x\}$ . *Acta Math. Hungar.* **118**, 155–170 (2008).
- [8] V. F. GAPOSHKIN: The central limit theorem for weakly dependent sequences. *Theory of Probability and Applications* **15**, 649–666 (1970).
- [9] I. A. IBRAGIMOV: Some limit theorems for stationary processes. *Theory of Probability and Applications* **7**, 349–382 (1962).
- [10] H. KESTEN: The discrepancy of random sequences  $\{kx\}$ , *Acta Arith.* **10**, 183–213 (1964/65).
- [11] A. KHINCHIN: Einige Sätze über Kettenbrüche mit Anwendungen auf die Theorie der Diophantischen Approximationen, *Math. Ann.* **92**, 115–125 (1923).
- [12] P. LÉVY: *Theorie de L'Addition des Variables Aleatoires*. Gauthier-Villars, Paris 1937.
- [13] A. RÉNYI: Representations for real numbers and their ergodic properties. *Acta Mathematica Academiae Scientiarum Hungaricae* **8**, 477–493 (1957)
- [14] M. ROSENBLATT: A central limit theorem and a strong mixing condition. *Proceedings of the National Academy of Sciences USA* **42**, 43–47 (1956).

- [15] R. SALEM and A. ZYGMUND: On lacunary trigonometric series. *Proceedings of the National Academy of Sciences USA* **33**, 333–338 (1947)
- [16] P. SCHATTE: On strong versions of the central limit theorem. *Mathematische Nachrichten* **137**, 249–256 (1988).
- [17] S. TAKAHASHI: A gap sequence with gaps bigger than the Hadamards. *Tôhoku Math. J.* **13**, 105–111 (1961).
- [18] S. TAKAHASHI: The law of the iterated logarithm for a gap sequence with infinite gaps. *Tôhoku Math. J.* **15**, 281–288 (1963).
- [19] A. WALFISZ: Ein metrischer Satz über diophantische Approximationen. *Fundamenta Mathematicae* **16**, 361–385 (1930).
- [20] H. WEYL: Über die Gleichverteilung von Zahlen mod eins. *Mathematische Annalen* **77**, 313–352 (1916).

### Walter Philipp's scientific publications

1. Ein metrischer Satz über die Gleichverteilung mod 1. *Arch. Math.* **12**, 429–433 (1961).
2. Über die Einfachheit von Funktionenalgebren. *Monatshefte für Mathematik* **66**, 441–451 (1962) (with Wilfried Nöbauer).
3. Über die Einfachheit von Funktionenalgebren über Verbänden. *Monatshefte für Mathematik* **67**, 259–268 (1963).
4. Die Einfachheit der mehrdimensionalen Funktionenalgebren. *Arch. Math.* **15**, 1–5 (1964) (with Wilfried Nöbauer).
5. Über einen Satz von Davenport-Erdős-LeVeque. *Monatshefte für Mathematik* **68**, 52–58 (1964).
6. An  $n$ -dimensional analogue of a theorem of H. Weyl. *Compositio Mathematicae* **16**, 161–163 (1964).
7. A note on a problem of Littlewood about Diophantine approximation. *Mathematika* **11**, 137–141 (1964).
8. Some metrical theorems in number theory. *Pacific Journal of Mathematics* **20**, 109–127 (1967).
9. Ein zentraler Grenzwertsatz mit Anwendungen auf die Zahlentheorie. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **8**, 185–203 (1967).
10. Das Gesetz vom iterierten Logarithmus für stark mischende stationäre Prozesse. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **8**, 204–209 (1967).
11. Mischungseigenschaften gewisser auf dem Torus definierter Endomorphismen. *Mathematische Zeitschrift* **101**, 369–374 (1967).
12. Das Gesetz vom iterierten Logarithmus mit Anwendungen auf die Zahlentheorie. *Mathematische Annalen* **180**, 75–94 (1969). (Corrigendum: *Mathematische Annalen* **190**, 338 (1970)).
13. The remainder in the central limit theorem for mixing stochastic processes. *Annals of Mathematical Statistics* **40**, 601–609 (1969).
14. Zwei Grenzwertsätze für Kettenbrüche. *Mathematische Annalen* **181**, 152 – 156 (1969) (with Olaf P. Stackelberg).
15. The central limit problem for mixing sequences of random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **12**, 155–171 (1969).
16. The law of the iterated logarithm for mixing stochastic processes. *Annals of Mathematical Statistics* **40**, 1985–1991 (1969).

17. Some metrical theorems in number theory II. *Duke Mathematical Journal* **37**, 447–458 (1970) (Errata: *Duke Mathematical Journal* **37**, 788 (1970)).
18. Mixing sequences of random variables and probabilistic number theory. *Memoirs of the American Mathematical Society* **114** (1971)
19. An invariance principle for mixing sequences of random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **25**, 223–237 (1972) (with Geoffrey R. Webb).
20. On a theorem of Erdős and Turán on uniform distribution. *Proceedings of the Number Theory Conference (University of Colorado, Boulder 1972)*, 180–182 (1972) (with Harald Niederreiter).
21. Berry-Esseen bounds and a theorem of Erdős and Turán on uniform distribution mod 1. *Duke Mathematical Journal* **40**, 633–649 (1973) (with Harald Niederreiter)
22. Empirical distribution functions and uniform distribution mod 1. *Diophantine Approximation and its applications* (Proceedings Conference Washington, D.C. 1972), Academic Press, New York, 211–234 (1973).
23. Arithmetic functions and Brownian motion. Analytic methods in number theory. *Proc. Sympos. Pure Math.* **XXIV**, 233–245, American Mathematical Society, Providence R.I. 1973.
24. Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arithmetica* **26**, 241–251 (1974)
25. Distances of probability measures and uniform distribution mod 1. *Mathematische Zeitschrift* **142**, 195–202 (1975) (with Rudolf Mück).
26. A conjecture of Erdős on continued fractions. *Acta Arithmetica* **28**, 379–386 (1975)
27. Asymptotic fluctuation behaviour of sums of weakly dependent random variables. *Limit theorems of probability theory (Colloq. Kesthely 1974)*, *Colloquia Math. Soc. Janos Bolyai* **11**, 273–296, North Holland Amsterdam 1975 (with William F. Stout).
28. Almost sure invariance principles for partial sums of weakly dependent random variables. *Memoir of the American Mathematical Society* **161** (1975) (with William F. Stout).
29. Almost sure invariance principles for empirical distribution functions of weakly dependent random variables. *Empirical distribution and processes, Lecture Notes in Mathematics* **566**, 83–105, Springer Verlag, Berlin 1976.
30. A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables. *Annals of Probability* **5**, 319–350 (1977).
31. An almost sure invariance principle for the empirical distribution function of mixing random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **41**, 115–137 (1977) (with István Berkes).
32. A uniform law of the iterated logarithm for classes of functions. *Annals of Probability* **6**, 930–952 (1978) (with Robert Kaufman).
33. An a.s. invariance principle for lacunary series  $f(n_k x)$ . *Acta Math. Acad. Sci. Hungar.* **34**, 141–155 (1979) (with István Berkes).
34. Approximation theorems for independent and weakly dependent random vectors. *Annals of Probability* **7**, 29–54 (1979) (with István Berkes).
35. Almost sure invariance principles for sums of  $B$ -valued random variables. *Probability in Banach spaces II (Proc. Second Internat. Conference, Oberwolfach 1978)*, *Lecture Notes in Mathematics* **709**, 171–193, Springer Verlag, Berlin 1979.
36. A new method to prove limit theorems for mixing random variables and its applications. *Transactions of the Eighth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes (Prague 1978)*, Vol. C, 39–46, Reidel, Dordrecht-Boston 1979 (with István Berkes).

37. Weak and  $L_p$ -invariance principles for sums of  $B$ -valued random variables. *Annals of Probability* **8**, 1003–1036 (1980) (Correction *Ann. Prob.***14**, 1095–1101 (1986)).
38. Almost sure approximation theorems for the multivariate empirical process. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **54**, 1–13 (1980). (with Laurence Pinzur)
39. Almost sure invariance principles for partial sums of mixing  $B$ -valued random variables. *Annals of Probability* **8**, 1003–1036 (1980) (with James Kuelbs).
40. Almost sure invariance principles for sums of  $B$ -valued random variables with applications to random Fourier series and the empirical characteristic process. *Transactions of the American Mathematical Society* **269**, 67–90 (1982) (with Michael B. Marcus).
41. Almost sure invariance principles for weakly dependent vector-valued random variables. *Annals of Probability Theory* **10**, 689–701 (1982) (with Herold Dehling).
42. An almost sure invariance principle for Hilbert space valued martingales. *Transactions of the American Mathematical Society* **273**, 231–251 (1982) (with Gregory Morrow).
43. Invariance Principles for sums of Banach space valued random elements and empirical processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **62**, 509–552 (1983) (with Richard M. Dudley).
44. An almost sure invariance principle for triangular arrays of Banach space valued random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **65**, 483–491 (1984) (with André Dabrowski and Herold Dehling).
45. Versik processes and very weak Bernoulli processes with summable rates are independent. *Proceedings of the American Mathematical Society* **91**, 618–624 (1984) (with Herold Dehling and Manfred Denker).
46. Invariance Principles for von Mises and  $U$ -Statistics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **67**, 439–467 (1984) (with Herold Dehling and Manfred Denker).
47. Approximation by Brownian motion of Gibbs measures and flows under a function. *Ergodic Theory and Dynamical Systems* **4**, 541–552 (1984) (with Manfred Denker).
48. Invariance Principles for sums of mixing random elements and the multivariate empirical process. *Limit theorems in probability and statistics, Veszprem 1982, Colloq. Math. Soc. Janos Bolyai* **36**, 843–873, North Holland, Amsterdam 1984.
49. A bounded law of the iterated logarithm for Hilbert space valued random variables and its application to  $U$ -statistics. *Probability Theory and Related Fields* **72**, 111–131 (1986) (with Herold Dehling and Manfred Denker).
50. Invariance principles for martingales and sums of independent random variables. *Mathematische Zeitschrift* **192**, 253–264 (1986) (with William F. Stout).
51. Invariance principles for partial sum processes and empirical processes indexed by sets. *Probability Theory and Related Fields* **73**, 11–42 (1986) (with Gregory Morrow).
52. A strong approximation theorem for sums of random vectors in the domain of attraction of a stable law. *Acta Mathematica Hungarica* **48**, 161–172 (1986) (with Istvan Berkes, Andre Dabrowski and Herold Dehling).
53. A note on the almost sure approximation of weakly dependent random variables. *Monatshefte für Mathematik* **102**, 227–236 (1986).
54. Central limit theorems for mixing sequences under minimal conditions. *Annals of Probability* **14**, 1359–1370 (1986) (with Herold Dehling and Manfred Denker).
55. Invariance principles for independent and weakly dependent random variables. *Dependence in Probability and Statistics (Oberwolfach 1985)*, Progress in Probability and Statistics **11**, 225–268, Birkhäuser Boston 1986.

56. The almost sure invariance principle for the empirical process of  $U$ -statistic structure. *Annales de l'Institut Henri Poincaré - Probability and Statistics* **23**, 121–134 (1987) (with Herold Dehling and Manfred Denker).
57. Limit theorems for sums of partial quotients of continued fractions. *Monatshefte für Mathematik* **105**, 195–206 (1988).
58. A note on the almost sure central limit theorem. *Statistics and Probability Letters* **9**, 201–205 (1990) (with Michael T. Lacey).
59. The problem of embedding vector-valued martingales in a Gaussian process. *Theory of Probability and its Applications* **35**, 374–377 (1990) (with Ditlef Monrad).
60. Nearby variables with nearby conditional laws and a strong approximation theorem for Hilbert space valued martingales. *Probability Theory and Related Fields* **88**, 381–404 (1991) (with Ditlef Monrad).
61. Conditional versions of the Strassen-Dudley theorem. *Probability in Banach spaces 8 (Brunswick, Maine 1991)*, Progress in Probability **30**, 116–127, Birkhäuser Boston 1992 (with Ditlef Monrad).
62. Empirical distribution functions and strong approximation theorems for dependent random variables. A problem of Baker in probabilistic number theory. *Transactions of the American Mathematical Society* **345**, 705–727 (1994).
63. The size of trigonometric and Walsh series and uniform distribution mod 1. *Journal of the London Mathematical Society (2)* **50**, 454–464 (1994) (with István Berkes).
64. Trigonometric series and uniform distribution mod 1. *Studia Sci. Math. Hungarica* **31**, 15–25 (1996) (with István Berkes).
65. Limit theorems for mixing sequences without rate assumptions. *Annals of Probability* **26**, 805–831 (1998) (with István Berkes).
66. A limit theorem for lacunary series  $\sum f(n_k x)$ . *Studia Sci. Math. Hungarica* **34**, 1–13 (1998) (with István Berkes).
67. Pair correlations and  $U$ -statistics for independent and weakly dependent random variables. *Illinois Journal of Mathematics* **45**, 559–580 (2001) (with István Berkes and Robert Tichy).
68. Metric theorems for distribution measures of pseudorandom sequences. Dedicated to Edmund Hlawka on the occasion of his 85th birthday. *Monatshefte für Mathematik* **135**, 321–326 (2002) (with Robert Tichy).
69. The profound impact of Paul Erdős on probability limit theory - a personal account. *Paul Erdős and his mathematics, Budapest 1999*, Bolyai Society Math. Stud. **11**, 549–566, Budapest 2002.
70. Empirical process techniques for dependent data. *Empirical process techniques for dependent data*, 3–113, Birkhäuser Boston 2002 (with Herold Dehling).
71. Empirical processes in probabilistic number theory; the LIL for the discrepancy of  $n_k \omega \bmod 1$ . *Illinois Journal of Mathematics* **50**, 107–145 (2006) (with István Berkes and Robert Tichy).
72. Pseudorandom numbers and entropy conditions. *J. of Complexity* **23**, 516–527 (2007) (with István Berkes and Robert Tichy).
73. Metric discrepancy results for sequences  $\{n_k x\}$  and diophantine equations. In: *Diophantine Approximations* (eds. H.P. Schlickewei, K. Schmidt and R. Tichy), 95–105. Springer, 2008 (with István Berkes and Robert Tichy).
74. Entropy conditions for subsequences of random variables with applications to empirical processes. *Monatshefte Math.* **153**, 183–204 (2008) (with István Berkes and Robert Tichy).

### **Walter Philipp's Ph.D. students**

1. Lawrence Pinzur (1978)
2. Gregory Morrow (1979)
3. Sompong Dhompongsa (1981)
4. Herold Dehling (1981, University of Göttingen)
5. André Dabrowski (1982)
6. Wolfram Strittmatter (1985, University of Freiburg)
7. Michael Lacey (1987)