New Bounds for Discrepancy Function and Small Ball Inequality in All Dimensions

Michael Lacey

Georgia Institute of Technology

We will address a question of relevance to Irregularities of Distribution, Approximation Theory, and Probability Theory. The most elementary formulation of the set of questions comes from Irregularities of Distribution. Given a collection P of N points in the unit cube in dimension d, define the Discrepancy Function

$$D_N(x) = |[0,x) \cap P| - N|[0,x)| \,, \qquad x \in [0,1]^d \,.$$

By $[0,x)$ we mean the d-dimensional rectangle anchored at the origin and at point x. The Discrepancy Function at each point x is the difference between the number of points of the collection P that are in the rectangle $[0,x)$, and number of points expected to be in the rectangle. A classical result of Klaus Roth shows that D_N can never be very small, regardless how cleverly the collection P is chosen. Namely, $\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$. This is a sharp estimate. The L^∞ norm of the Discrepancy Function is conjectured to be larger. In two dimensions, one has the definitive bound of Wolfgang Schmidt, $\|D_N\|_\infty \gtrsim \log N$. We will discuss an extension of this result to arbitrary dimensions: For dimensions $d \geq 3$, there is an $\epsilon(d) > 0$ so that we have the universal estimate

$$\|D_N\|_\infty \gtrsim (\log N)^{(d-1)/2+\epsilon(d)} \,.$$

That is, the gain over the Roth bound is $(\log N)^{\epsilon(d)}$. This result extends a result of Jozef Beck, in dimension $d = 3$, which gave a doubly-logarithmic improvement of the Roth bound. The connection to Approximation Theory and Probability Theory is made through the so-called Small Ball Inequality, a non-trivial lower bound on the L^∞ norm of sums of Haar functions in higher dimensions, supported on rectangles of a fixed volume. In this context, we extend results of Talagrand to dimensions $d \geq 3$. Joint work with Dmitriy Bilyk and Armen Vagharshakyan.