Proving a limit theorem for the continued fraction expansion using an iterated function system with a strictly stationary iteration mechanism

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Let Ω be the set of irrationals in $I = [0,1]$ and let $[a_1(\omega), a_2(\omega), \cdots]$ denote the continued fraction expansion of $\omega \in \Omega$. The sequence $(a_n)_{n \in \mathbf{N}_+}$, $\mathbf{N}_+ = \{1, 2, \dots\}$, is defined a.e. in I and is strictly stationary on the probability space $(I, \mathcal{B}_I, \gamma)$, where $\mathcal{B}_I$ is the collection of Borel subsets of I and γ is Gauss' measure on $\mathcal{B}_I$ defined by

$$\gamma(A) = \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{x+1}, \ A \in \mathcal{B}_I.$$

Cf. ([IK], Section 1.2).

Define extended incomplete quotients $\overline{a}_\ell$, $\ell \in \mathbf{Z} = \{\cdots, -1, 0, 1, \cdots\}$, by $\overline{a}_\ell(\omega,\theta) = a_\ell(\omega), \overline{a}_0(\omega,\theta) = a_1(\theta), \overline{a}_{-\ell}(\omega,\theta) = a_{\ell+1}(\theta)$ for $\ell \in \mathbf{N}_+$ and $(\omega,\theta) = \Omega^2$. The doubly infinite sequence $(\overline{a}_\ell)_{\ell \in \mathbf{Z}}$ is defined a.e. in I^2 and is strictly stationary on the probability space $\left(I^2, \mathcal{B}_{I^2}, \overline{\gamma}\right)$, where $\mathcal{B}_{I^2}$ is the collection of Borel subsets of I^2 and $\overline{\gamma}$ is the extended Gauss measure on $\mathcal{B}_{I^2}$ defined by

$$\overline{\gamma}(B) = \frac{1}{\log 2} \int\int_B \frac{\mathrm{d}x\mathrm{d}y}{(xy+1)^2}, \ B \in \mathcal{B}_{I^2}.$$

Cf. ([IK], Subsections 1.3.1 and 1.3.3).

Set $\overline{s}_\ell = [\overline{a}_\ell, , \overline{a}_{\ell-1}, \dots]$, $\ell \in \mathbf{Z}$, and $s_0^a = a$, $s_{n+1}^a = 1/\left(a_{n+1} + s_n^a\right)$, $a \in I$, $n \in \mathbf{N} = \{0\} \cup \mathbf{N}_+$. We prove that for any $a \in I$ the sequence $\left(\ s_n^a,\ s_{n+1}^a, \dots\right)$ on $(I, \mathcal{B}_I, \gamma)$ converges in distribution as $n \to \infty$ to the strictly stationary sequence $(\overline{s}_0, \overline{s}_1 \cdots)$ on $\left(I^2, \mathcal{B}_{I^2}, \overline{\gamma}\right)$. In particular,

$$\lim_{n\to\infty} \gamma\left(s_n^a < x\right) = \gamma([0,x]), \ x \in I,$$

for any $a \in I$, cf. ([IK], Subsection 2.5.3), and

$$\lim_{n\to\infty} \gamma\left(s_n^a < x,\ s_{n+1}^a < y\right) = \lim_{n\to\infty} \overline{\gamma}\left(\overline{s}_0 < x,\ \overline{s}_1 < y\right) = \lim_{n\to\infty} \overline{\gamma}\left(\overline{s}_{-1} < x,\ \overline{s}_0 < y\right)$$

$$= \begin{cases} \dfrac{1}{\log 2} \log\left(1 + \dfrac{x}{[1/y]+1}\right) & \text{if } \ x \le \{1/y\} \\[2em] \dfrac{1}{\log 2} \log \dfrac{(y+1)([1/y]+x)}{[1/y]+1} & \text{if } \ x > \{1/y\} \end{cases}$$

for any $a \in I$ and $(x, y) \in I^2$. Here, $[\cdot]$ and $\{\cdot\}$ stand for whole part and fractionary part, respectively.

In the proof we use Kingman's subadditive ergodic theorem (see [K]) and work of J.H. Elton [E] for iterated function systems with Lipschitz self-mappings obeying a strictly stationary mechanism instead of an i.i.d. one.

References

[E] J.H. Elton, A multiplicative ergodic theorem for Lipschitz maps. *Stochastic Process. Appl.* **34**(1990), 39-47.

[IK] M. Iosifescu and C. Kraaikamp, *Metrical Theory of Continued Fractions.* Kluwer, Dordrecht, 2002.

[K] U. Krengel, *Ergodic Theorems.* With a supplement by Antoine Brunel. Walter de Gruyter, Berlin, 1985.