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On the Transmuted Additive Weibull Distribution

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Abstract: In this article a continuous distribution, the so-called transmuted additive Weibull distribution, that extends the additive Weibull distribution and some other distributions is proposed and studied. We will use the quadratic rank transmutation map proposed by Shaw and Buckley (2009) in order to generate the transmuted additive Weibull distribution. Various structural properties of the new distribution including explicit expressions for the moments, random number generation and order statistics are derived. Maximum likelihood estimation of the unknown parameters of the new model for complete sample is also discussed. It will be shown that the analytical results are applicable to model real world data.

Zusammenfassung: In diesem Artikel wird eine stetige Verteilung vorgeschlagen und untersucht, die sogenannte additive umgewandelte Weibull-Verteilung, welche die additive Weibull-Verteilung und einige andere Verteilungen erweitert. Wir verwenden die quadratische Rang Transmutations-Abbildung, vorgeschlagen in Shaw and Buckley (2009), um die additive umgewandelte Weibull-Verteilung zu erzeugen. Verschiedene strukturelle Eigenschaften der neuen Verteilung einschließlich explizite Ausdrücke für die Momente, Erzeugung von Zufallszahlen, und Ordnungsstatistiken werden hergeleitet. Maximum Likelihood Schätzung der unbekannten Parameter dieses neuen Modells für die vollständige Stichprobe wird ebenfalls diskutiert. Es wird gezeigt, dass diese analytischen Ergebnisse verwendbar sind um reale Daten zu modellieren.

Keywords: Additive Weibull Distribution, Order Statistics, Transmutation Map, Maximum Likelihood Estimation, Reliability Function.

1 Introduction

Since the quality of the procedures used in statistical analysis depends heavily on the assumed probability model or distributions, a significant contribution has been made on generalization of some well-known distributions and their successful applications. For many mechanical and electronic components, the hazard (failure) rate function has a bathtub shape. It is well-known that, because of design and manufacturing problems, the failure rate is high at the beginning of a product life cycle and decreases toward a constant level. After reaching a certain age, the product enters the wear-out phase and the failure rate starts to increase. Despite the fact that this phenomenon has been presented in many reliability engineering texts, few practical models possessing this property have appeared in the literature. Because of this, only a part of the bathtub curve is considered at any one time. Another common fact is that most engineers may be interested only in a part of the lifetime, because at component level, they only see one part of the failure rate function. However, it will be helpful to have a model that is reasonably simple and good for the whole product life cycle for making overall decisions. Furthermore, for complex systems, both the decreasing and increasing parts of the failure rate fall into the ordinary product lifetime. Lifetime distributions for many components usually have a bathtub-shaped failure rate in practice. However, there are very few standard distributions to model this type of failure rate function. The Weibull distribution has been used in many different fields with many applications, see for example Lai, Xie, and Murthy (2003). The hazard function of the Weibull distribution can only be increasing, decreasing or constant.Thus it can not be used to model lifetime data with a bathtub shaped hazard function, such as human mortality and machine life cycles. For many years, researchers have been developing various extensions and modified forms of the Weibull distribution, with different number of parameters. A state-of-the-art survey on the class of such distributions can be found in Lai, Xie, and Murthy (2001) and Nadarajah (2009). Xie and Lai (1995) proposed a four-parameter additive Weibull (AW) distribution as a competitive model. A random variable X is said to have a AW distribution if its cumulative distribution function (cdf) is

$$F(x) = 1 - e^{-\alpha x^{\theta} - \gamma x^{\beta}}\,, \qquad x \geq 0\,, \tag{1}$$

where α, θ, γ, and β are non-negative, with $\theta < 1 < \beta$ (or $\beta < 1 < \theta$). Note that θ and β are the shape parameters and α and γ are scale parameters.

The probability density function (pdf) of the AW distribution is

$$f(x) = \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) e^{-\alpha x^{\theta} - \gamma x^{\beta}}\,, \tag{2}$$

and the hazard rate function is given by

$$\begin{aligned} h(x) &= \frac{f(x)}{1 - F(x)} \\ &= \frac{\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) e^{-\alpha x^{\theta} - \gamma x^{\beta}}}{e^{-\alpha x^{\theta} - \gamma x^{\beta}}} \\ &= \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)\,. \end{aligned}$$

The change point x where the hazard rate function $h(x)$ achieves its minimum is at

$$x = \left(\frac{\theta(1-\theta)\alpha}{\beta(\beta-1)\gamma}\right)^{\frac{1}{\beta-\theta}}\,,$$

when $\theta < 1 < \beta$. It is important to note that the change point remains the same when $\beta < 1 < \theta$. In this article the four parameter AW distribution is embedded in a larger family obtained by introducing an additional parameter. We will call the generalized distribution as the transmuted additive Weibull (TAW) distribution.

The rest of the article is organized as follows. In Section 2 we present the expression of the pdf and cdf of the subject distribution and some special sub-models. In Section 3 we study the statistical properties including quantile functions, moments, moment generating function etc. The reliability functions of the subject model are given in Section 4. The minimum, maximum and median order statistics models are discussed in Section 5. In Section 6 we demonstrate the maximum likelihood estimates and the asymptotic confidence intervals of the unknown parameters. Finally, in Section 7 we present a real world data analysis to illustrate the usefulness of the proposed distribution.

2 Transmuted Additive Weibull Distribution

A random variable X is said to have a transmuted probability distribution with cdf $F(x)$, if

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2\,, \qquad |\lambda| \leq 1\,,$$

where $G(x)$ is the cdf of the base distribution. Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal and Tsokos (2011) studied the TW as a generalization of Weibull distribution. Khan and King (2013) extended the MW to a TMW distribution.

In this section we present the TAW distribution and the sub-models of this distribution. Now, using (1) and (2) we have the cdf of the TAW distribution

$$F_{TAW}(x) = \left(1 - e^{-\alpha x^\theta - \gamma x^\beta}\right)\left(1 + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)\,, \tag{3}$$

where θ and β are the shape parameters representing the different patterns of the TAW distribution and are positive, α and γ are scale parameters representing the characteristic life and are also positive, and λ is the transmuted parameter. The probability density function (pdf) of a TAW distribution is given by

$$f_{TAW}(x) = \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)e^{-\alpha x^\theta - \gamma x^\beta}\left(1 - \lambda + 2\lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)\,. \tag{4}$$

The TAW distribution is a very flexible model that approaches to different distributions when its parameters vary. The flexibility of the TAW distribution is explained in Table 1.

Table 1: Transmuted additive Weibull distribution and some of its sub-models

Model	α	θ	γ	β	λ	Cumulative distribution function
TAW	–	–	–	–	–	$(1-\exp(-\alpha x^\theta - \gamma x^\beta))(1+\lambda\exp(-\alpha x^\theta - \gamma x^\beta))$
TMW	–	1	–	–	–	$(1-\exp(-\alpha x - \gamma x^\beta))(1+\lambda\exp(-\alpha x - \gamma x^\beta))$
TLF	–	1	–	2	–	$(1-\exp(-\alpha x - \gamma x^2))(1+\lambda\exp(-\alpha x - \gamma x^2))$
TME	–	1	–	1	–	$(1-\exp(-\alpha x - \gamma x))(1+\lambda\exp -\alpha x - \gamma x))$
AW	–	–	–	–	0	$1-\exp(-\alpha x^\theta - \gamma x^\beta)$
MW	–	1	–	–	0	$1-\exp(-\alpha x - \gamma x^\beta)$
MR	–	1	–	2	0	$1-\exp(-\alpha x - \gamma x^2)$
ME	–	1	–	1	0	$1-\exp(-\alpha x - \gamma x)$
TW	0	–	–	–	–	$(1-\exp(-\gamma x^\beta))(1+\lambda\exp(-\gamma x^\beta))$
TR	0	–	–	2	–	$(1-\exp(-\gamma x^2))(1+\lambda\exp(-\gamma x^2))$
TE	0	–	–	1	–	$(1-\exp(-\gamma x))(1+\lambda\exp -\gamma x))$
W	0	–	–	–	0	$1-\exp(-\gamma x^\beta)$
R	0	–	–	2	0	$1-\exp(-\gamma x^2)$
E	0	–	–	1	0	$1-\exp(-\gamma x)$

Abbriviations: T = Transmuted, A = Additive, M = Modified, W = Weibull, E = Exponential, LF = Linear Failure, R = Rayleigh.

The subject distribution includes as special cases the transmuted modified Weibull (TMW), the transmuted modified exponential (TME), transmuted linear failure rate (TLFR), transmuted Weibull (TW), transmuted Rayleigh (TR) and transmuted exponential, Weibull, Rayleigh and exponential distributions.

Figure 1 shows some of the the sub-models of the TAW distribution that approach several different lifetime distributions for particular choices of its parameters. Figure 2 illustrates the graphical behavior of the pdf of TAW distribution for selected values of the parameters. Note that the figure on the left exhibits the shape of the distribution for different choice of all the parameters whereas the figure on the right exhibits the behavior shape as λ varies from -1 to 1 while keeping all other four parameters fixed.

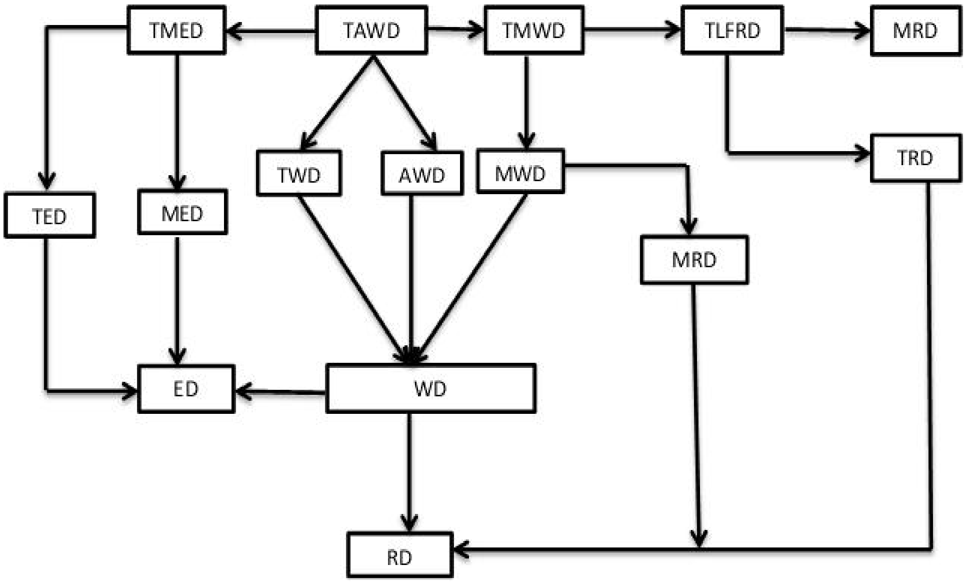

Figure 1: Sub-models of transmuted additive Weibull distributions

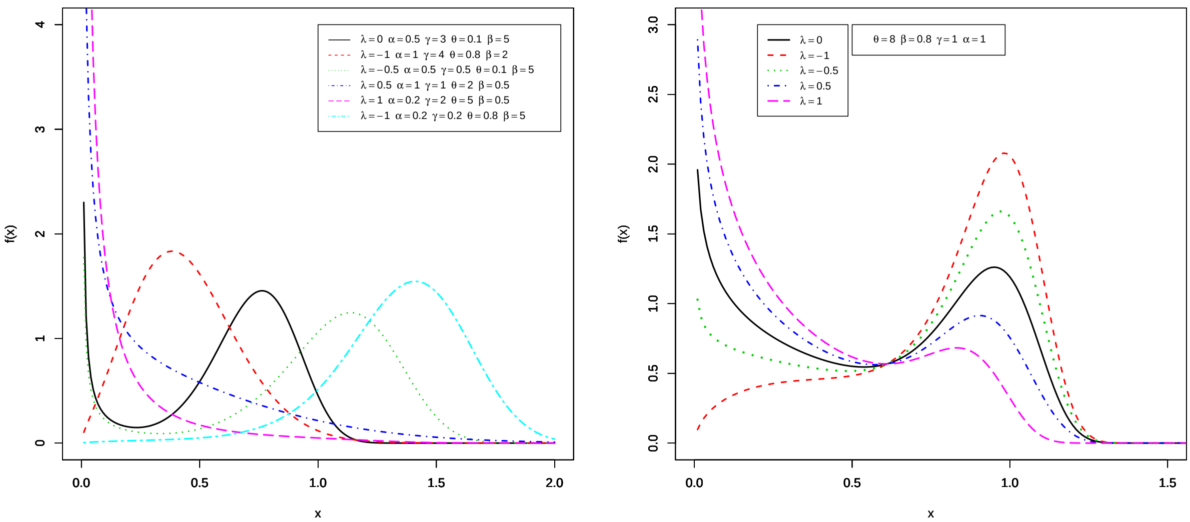

Figure 2: Density functions of various transmuted additive Weibull distributions

3 Statistical Properties

In this section we discuss few statistical properties of the TAW distribution.

3.1 Quantiles

The quantile x_q of the $T_{AWD}(\alpha,\beta,\theta,\gamma,\lambda,x)$ is the real solution of the equation

$$\alpha x_q^{\theta}+\gamma x_q^{\beta}+\log\left(1-\frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4\lambda q}}{2\lambda}\right)=0\,. \tag{5}$$

Since the above equation has no closed form solution in x_q, we have to use numerical methods to get the quantiles.

We can use (5) to derive quantiles for some special cases of the TAW distribution. Let

$$h(\lambda,q)=\log\left(1-\frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4\lambda q}}{2\lambda}\right),$$

then the following results can be derived.
(i) Setting $\theta=1$, the x_q quantile of the $T_{MW}(\alpha,\beta,\gamma,\lambda,x)$ distribution is the solution of

$$\alpha x_q+\gamma x_q^{\beta}+h(\lambda,q)=0\,.$$

(ii) Setting $\theta=1$ and $\beta=2$, the x_q quantile of the $T_{LFRD}(\alpha,\gamma,\lambda,x)$ distribution is

$$x_q=\frac{-\alpha+\sqrt{\alpha^2-4\gamma h(\lambda,q)}}{2\gamma}\,.$$

(iii) Setting $\theta=1$ and $\alpha=0$, the x_q quantile of the $T_{WD}(\beta,\gamma,\lambda,x)$ distribution is

$$x_q=\left(-\frac{1}{\gamma}h(\lambda,q)\right)^{\frac{1}{\beta}}\,.$$

(iv) Setting $\theta=1$, $\alpha=0$, and $\beta=2$, the x_q quantile of the $T_{RD}(\alpha,\gamma,\lambda,x)$ distribution is

$$x_q=\sqrt{-\frac{1}{\gamma}h(\lambda,q)}\,.$$

(v) Setting $\theta=1$, $\alpha=0$, and $\beta=1$, the x_q quantile of the $T_{ED}(\alpha,\gamma,\lambda,x)$ distribution is

$$x_q=-\frac{1}{\gamma}h(\lambda,q)\,.$$

In all the above cases the median can be obtained by setting $q=0.5$.

3.2 Random Number Generation

In order to generate random numbers from a $T_{AWD}(\alpha,\beta,\theta,\gamma,\lambda,x)$ distribution we need to solve

$$\left[1-\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)\right]\left[1+\lambda\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)\right]=u\,,\qquad \text{where}\quad u\sim U(0,1)\,.$$

This yields,

$$\alpha x^{\theta}+\gamma x^{\beta}+h(\lambda,u)=0\,, \tag{6}$$

where

$$h(\lambda,u)=\log\left(1-\frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda}\right)\,.$$

Equation (6) does not have a closed form solution so we generate u from U(0, 1) and solve for x in order to generate random numbers from a TAW distribution.

3.3 Moments

The rth order moment $\mu_r=\mathrm{E}(X^r)$ of a TAW distribution is given by Theorem 3.1 below.

Theorem 3.1 If X is from a $T_{AWD}(\alpha,\beta,\theta,\gamma,\lambda,x)$ distribution with $|\lambda|\le 1$, then the rth moment of X is given by

$$\begin{aligned}\mu_r=&\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\gamma^k}{\alpha^{\frac{r+k\beta}{\theta}}}\left[(1-\lambda)+\lambda\frac{2^k}{2^{\frac{r+k\beta}{\theta}}}\right]\Gamma\left(\frac{r+k\beta+\theta}{\theta}\right)\\&+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\alpha^k}{\gamma^{\frac{r+k\theta}{\beta}}}\left[(1-\lambda)+\lambda\frac{2^k}{2^{\frac{r+k\theta}{\beta}}}\right]\Gamma\left(\frac{r+k\theta+\beta}{\beta}\right)\,,\end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function, i.e.

$$\Gamma(a)=\int_0^{\infty}t^{a-1}\exp(-t)dt\,.$$

Proof:

$$\begin{aligned}\mu_r&=\int_0^{\infty}x^r f_{TAW}(\alpha,\beta,\theta,\gamma,\lambda,x)dx\\&=\int_0^{\infty}x^r\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\alpha x^{\theta}-\gamma x^{\beta}}\left(1-\lambda+2\lambda e^{-\alpha x^{\theta}-\gamma x^{\beta}}\right)dx\\&=(1-\lambda)\int_0^{\infty}\left(\alpha\theta x^{r+\theta-1}+\gamma\beta x^{r+\beta-1}\right)e^{-\alpha x^{\theta}-\gamma x^{\beta}}dx\\&\quad+2\lambda\int_0^{\infty}\left(\alpha\theta x^{r+\theta-1}+\gamma\beta x^{r+\beta-1}\right)e^{-2\alpha x^{\theta}-2\gamma x^{\beta}}dx\\&=(1-\lambda)I_1+2\lambda I_2\,.\end{aligned} \tag{7}$$

Now, using

$$e^{-\gamma x^{\beta}} = \sum_{k=0}^{\infty} \frac{(-1)^k \gamma^k x^{k\beta}}{k!}\,, \qquad e^{-\alpha x^{\theta}} = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k x^{k\theta}}{k!}\,,$$

we have

$$\begin{aligned} I_1 &= \int_0^{\infty} \left(\alpha\theta x^{r+\theta-1} + \gamma\beta x^{r+\beta-1}\right) e^{-\alpha x^{\theta}-\gamma x^{\beta}} dx \\ &= \int_0^{\infty} \alpha\theta x^{r+\theta-1} e^{-\alpha x^{\theta}} e^{-\gamma x^{\beta}} dx + \int_0^{\infty} \gamma\beta x^{r+\beta-1} e^{-\alpha x^{\theta}} e^{-\gamma x^{\beta}} dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\gamma^k}{\alpha^{\frac{r+k\beta}{\theta}}} \Gamma\left(\frac{r+k\beta+\theta}{\theta}\right) + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\alpha^k}{\gamma^{\frac{r+i\theta}{\beta}}} \Gamma\left(\frac{r+k\theta+\beta}{\beta}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[\frac{\gamma^k}{\alpha^{\frac{r+k\beta}{\theta}}} \Gamma\left(\frac{r+k\beta+\theta}{\theta}\right) + \frac{\alpha^k}{\gamma^{\frac{r+k\theta}{\beta}}} \Gamma\left(\frac{r+k\theta+\beta}{\beta}\right)\right]. \end{aligned} \tag{8}$$

Similarly, for I_2 we get

$$I_2 = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[\frac{(2\gamma)^k}{(2\alpha)^{\frac{r+k\beta}{\theta}}} \Gamma\left(\frac{r+k\beta+\theta}{\theta}\right) + \frac{(2\alpha)^k}{(2\gamma)^{\frac{r+k\theta}{\beta}}} \Gamma\left(\frac{r+k\theta+\beta}{\beta}\right)\right]. \tag{9}$$

Substituting (8) and (9) in (7) we get

$$\begin{aligned} \mu_r &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\gamma^k}{\alpha^{\frac{r+k\beta}{\theta}}} \left[(1-\lambda) + \lambda \frac{2^k}{2^{\frac{r+k\beta}{\theta}}}\right] \Gamma\left(\frac{r+k\beta+\theta}{\theta}\right) \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\alpha^k}{\gamma^{\frac{r+k\theta}{\beta}}} \left[(1-\lambda) + \lambda \frac{2^k}{2^{\frac{r+k\theta}{\beta}}}\right] \Gamma\left(\frac{r+k\theta+\beta}{\beta}\right). \end{aligned}$$

This completes the proof.

Note that in particular, if $\alpha = 0$, we have

$$\mu_r = \gamma^{\frac{-r}{\beta}} \left[(1-\lambda) + \lambda 2^{\frac{-r}{\beta}}\right] \Gamma\left(\frac{r}{\beta} + 1\right).$$

Similarly, if $\gamma = 0$, we get

$$\begin{aligned} \mu_r &= \int_0^{\infty} \alpha\theta x^{r+\theta-1} e^{-\alpha x^{\theta}} \left(1 - \lambda + 2\lambda e^{-\alpha x^{\theta}}\right) dx \\ &= \alpha^{\frac{-r}{\theta}} \left[(1-\lambda) + \lambda 2^{\frac{-r}{\theta}}\right] \Gamma\left(\frac{r}{\theta} + 1\right). \end{aligned}$$

3.4 Moment Generating Function

The moment generating function (mgf) of the TAW distribution is given by Theorem 3.2.

Theorem 3.2 If X is from a $T_{AW}(\alpha,\theta,\gamma,\beta,\lambda,x)$ distribution with $|\lambda| \leq 1$, then its mgf is

$$\begin{aligned} M_X(t) = &\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{t^j}{j!}(1-\lambda)\left[\gamma^k\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{\alpha^{\frac{j+k\beta}{\theta}}}+\alpha^k\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{\gamma^{\frac{j+k\theta}{\beta}}}\right] \\ &+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}\frac{t^j}{j!}\lambda\left[\gamma^k\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{(2\alpha)^{\frac{j+k\beta}{\theta}}}+\alpha^k\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{(2\gamma)^{\frac{j+k\theta}{\beta}}}\right]. \end{aligned}$$

Proof:

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} f_{TAW}(\alpha,\beta,\theta,\gamma,\lambda,x)dx \\ &= \int_0^\infty e^{tx}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\alpha x^\theta-\gamma x^\beta}\left(1-\lambda+2\lambda e^{-\alpha x^\theta-\gamma x^\beta}\right)dx \\ &= (1-\lambda)\int_0^\infty e^{tx}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\alpha x^\theta-\gamma x^\beta}dx \\ &\quad +2\lambda\int_0^\infty e^{tx}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-2\alpha x^\theta-2\gamma x^\beta}dx \\ &= (1-\lambda)I_1+2\lambda I_2\,. \end{aligned} \tag{10}$$

We have

$$\begin{aligned} I_1 &= \int_0^\infty e^{tx}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)e^{-\alpha x^\theta-\gamma x^\beta}dx \\ &= \int_0^\infty \alpha\theta x^{\theta-1}e^{tx}e^{-\alpha x^\theta}e^{-\gamma x^\beta}dx+\int_0^\infty \gamma\beta x^{\beta-1}e^{tx}e^{-\alpha x^\theta}e^{-\gamma x^\beta}dx \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k t^j\gamma^k}{j!k!}\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{\alpha^{\frac{j+k\beta}{\theta}}}+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k t^j\alpha^k}{j!k!}\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{\gamma^{\frac{j+k\theta}{\beta}}} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{t^j}{j!}\left[\gamma^k\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{\alpha^{\frac{j+k\beta}{\theta}}}+\alpha^k\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{\gamma^{\frac{j+k\theta}{\beta}}}\right]. \end{aligned} \tag{11}$$

Similarly, for I_2 we get

$$I_2 = \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}\frac{t^j}{j!}\left[\gamma^k\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{(2\alpha)^{\frac{j+k\beta}{\theta}}}+\alpha^k\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{(2\gamma)^{\frac{j+k\theta}{\beta}}}\right]. \tag{12}$$

Now, substituting (11) and (12) in (10) we get

$$M_X(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{t^j}{j!}(1-\lambda)\left[\gamma^k\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{\alpha^{\frac{j+k\beta}{\theta}}}+\alpha^k\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{\gamma^{\frac{j+k\theta}{\beta}}}\right]$$
$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}\frac{t^j}{j!}\lambda\left[\gamma^k\frac{\Gamma\left(\frac{j+k\beta+\theta}{\theta}\right)}{(2\alpha)^{\frac{j+k\beta}{\theta}}}+\alpha^k\frac{\Gamma\left(\frac{j+k\theta+\beta}{\beta}\right)}{(2\gamma)^{\frac{j+k\theta}{\beta}}}\right].$$

This completes the proof.

In particular, for $\alpha = 0$, we have

$$M_X(t) = \sum_{j=0}^{\infty}\frac{t^j}{j!}\gamma^{-\frac{j}{\beta}}\left[1-\lambda+\lambda 2^{-\frac{j}{\beta}}\right]\Gamma\left(\frac{j}{\beta}+1\right)$$

and for $\gamma = 0$, we get

$$M_X(t) = \sum_{j=0}^{\infty}\frac{t^j}{j!}\alpha^{-\frac{j}{\theta}}\left[1-\lambda+\lambda 2^{-\frac{j}{\theta}}\right]\Gamma\left(\frac{j}{\theta}+1\right).$$

4 Reliability Analysis

Because of the analytical structure of the TAW distribution, it can be a useful model to characterize failure time of a system. The reliability function also known as survival function of the TAW distribution is denoted by $R_{TAW}(t)$ and is given as

$$\begin{aligned} R_{TAW}(t) &= 1 - F_{TAW}(t) \\ &= 1-\left(1-e^{-\alpha t^{\theta}-\gamma t^{\beta}}\right)\left[1+\lambda e^{-\alpha t^{\theta}-\gamma t^{\beta}}\right) \\ &= e^{-\alpha t^{\theta}-\gamma t^{\beta}}\left(1-\lambda+\lambda e^{-\alpha t^{\theta}-\gamma t^{\beta}}\right). \end{aligned}$$

One of the most important quantities characterizing life phenomenon in life testing analysis is the hazard rate function defined by

$$h(t) = \frac{f(t)}{1-F(t)}.$$

The hazard rate function for a TAW distribution is given by

$$h_{TAW}(t) = \frac{\left(\alpha\theta t^{\theta-1}+\gamma\beta t^{\beta-1}\right)\left(1-\lambda+2\lambda e^{-\alpha t^{\theta}-\gamma t^{\beta}}\right)}{1-\lambda+\lambda e^{-\alpha t^{\theta}-\gamma t^{\beta}}}. \tag{13}$$

It is important to note that the unit for $h_{TAW}(t)$ is the probability of failure per unit of time, distance or cycles. The failure rates for several different distribution can be

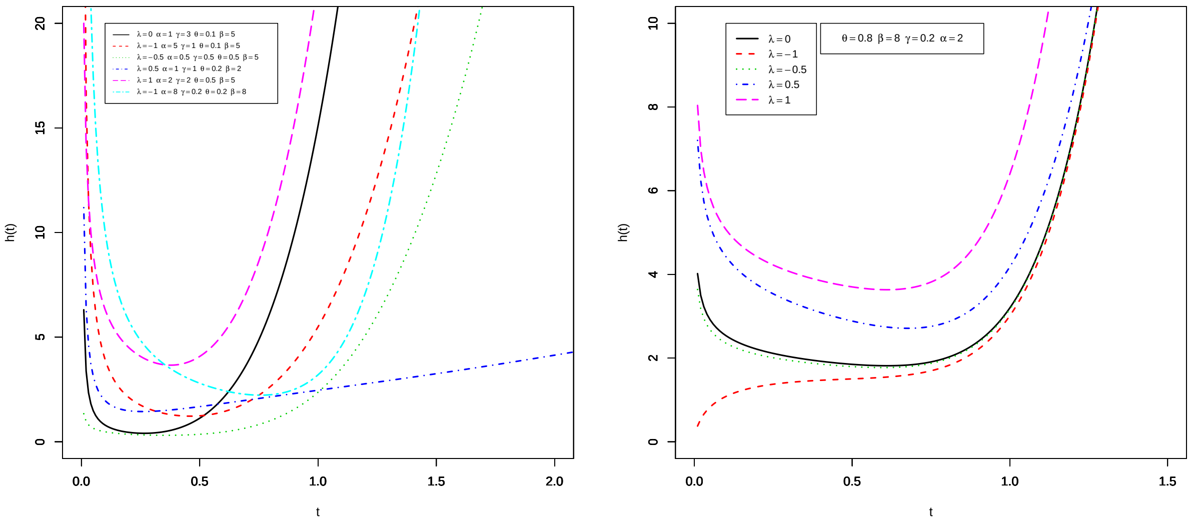

Figure 3: Hazard rate functions of various transmuted additive Weibull distributions

obtained by simply changing the parameters. Figure 3 illustrates the graphical behavior of the hazard rate function of the TAW distribution for selected values of the parameters.

Note that the figure on the left displays the hazard rate function for different choice of all parameters whereas the figure on the right exhibits the behavior of the hazard rate function as λ varies from -1 to 1 while keeping all other four parameters fixed.

In Table 2 we summarize few special cases of the hazard rate function of the TAW distribution (13). In addition to the cases presented in Table 2 we can also easily obtain hazard rate functions of the Weibull, Rayleigh and exponential distribution from the TAW hazard rate function. Also it should be noted that the reliability behavior remains the same when we consider the case $\beta < 1 < \theta$.

The cumulative hazard function, which describes how the risk of a particular outcome changes with time, is given by

$$\begin{aligned} H_{TAW}(t) &= \int_0^t h_{TAW}(x)dx \\ &= -\log\left[e^{-\alpha t^\theta - \gamma t^\beta}\left(1-\lambda+\lambda e^{-\alpha t^\theta - \gamma t^\beta}\right)\right] \\ &= \alpha t^\theta + \gamma t^\beta - \log\left(1-\lambda+\lambda e^{-\alpha t^\theta - \gamma t^\beta}\right) . \end{aligned}$$

Notice that the unit for $H_{TAW}(t)$ is the cumulative probability of failure per unit of time, distance or cycles. It describes how the risk of a particular outcome changes with time for a TAW distribution.

Table 2: Hazard rates of some sub-models of the transmuted additive Weibull distribution

Model	Parameters	Hazard rate function
TMWD	$\alpha, \gamma, \beta, \lambda$	$\dfrac{(\alpha+\gamma\beta t^{\beta-1})(1-\lambda+2\lambda e^{-\alpha t-\gamma t^{\beta}})}{1-\lambda+\lambda e^{-\alpha t-\gamma t^{\beta}}}$
AWD	$\alpha, \theta, \gamma, \beta$	$\alpha\theta t^{\theta-1}+\gamma\beta t^{\beta-1}$
TLFRD	α, γ, λ	$\dfrac{(\alpha+2\gamma t)(1-\lambda+2\lambda e^{-\alpha t-\gamma t^{2}})}{1-\lambda+\lambda e^{-\alpha t-\gamma t^{2}}}$
TMED	α, γ, λ	$\dfrac{(\alpha+\gamma)(1-\lambda+2\lambda e^{-\alpha t-\gamma t})}{1-\lambda+\lambda e^{-\alpha t-\gamma t}}$
TWD	γ, β, λ	$\dfrac{(\gamma\beta t^{\beta-1})(1-\lambda+2\lambda e^{-\gamma t^{\beta}})}{1-\lambda+\lambda e^{-\gamma t^{\beta}}}$
TED	γ, λ	$\dfrac{\gamma(1-\lambda+2\lambda e^{-\gamma t})}{1-\lambda+\lambda e^{-\gamma t}}$
TRD	γ, λ	$\dfrac{2\gamma t(1-\lambda+2\lambda e^{-\gamma t^{2}})}{1-\lambda+\lambda e^{-\gamma t^{2}}}$
MWD	α, γ, β	$\alpha+\gamma\beta t^{\beta-1}$
MRD	α, γ	$\alpha+2\gamma t$
MED	α, γ	$\alpha+\gamma$

5 Order Statistics

Order statistics are among the most fundamental tools in non-parametric statistics and statistical inference. These statistics also arise in the study of reliability and life testing. Some distributional properties of the maximum and the minimum of random variables have been extensively studied in the literature.

Let $X_1, \ldots, X_n$ be a simple random sample from a $T_{AWD}(\alpha, \beta, \theta, \gamma, \lambda, x)$ distribution with cdf and pdf as in (3) and (4), respectively. Let $X_{(1:n)} \leq \cdots \leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ denotes the lifetime of an "$(n-i+1)$ out of n" system which consists of n independent and identically distributed components. If $i = 1$ or $i = n$, such systems are better known as series or parallel systems, respectively. Considerable attention has been given to establish several reliability properties of such systems. It is well known that the cdf and pdf of $X_{(i:n)}$ for $i = 1, \ldots, n$ are given by

$$\begin{aligned} F_{i:n}(x) &= \sum_{k=i}^{n} \binom{n}{k} (F(x))^{k} (1-F(x))^{n-k} \\ &= \int_{0}^{F(x)} \frac{n!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i} dt \end{aligned}$$

and

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1-F(x))^{n-i} f(x) .$$

Define the minimum $X_{(1)} = \min(X_1, \dots, X_n)$, the maximum $X_{(n)} = \max(X_1, \dots, X_n)$, and the median as $X_{(m+1)}$ with $m = \lfloor n/2 \rfloor$.

5.1 Distribution of Minimum, Maximum and Median

Let $X_1, \dots, X_n$ denote a random sample from a TAW distribution, then the pdf of the the ith order statistic is given by

$$\begin{aligned} f_{i:n}(x) = {} & \frac{n!}{(i-1)!(n-i)!} \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) e^{-(n-i+1)(\alpha x^\theta + \gamma x^\beta)} \left(1 - e^{-\alpha x^\theta - \gamma x^\beta}\right)^{i-1} \\ & \times \left(1 + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)^{i-1} \left(1 - \lambda + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)^{n-i} \left(1 - \lambda + 2\lambda e^{-\alpha x^\theta - \gamma x^\beta}\right) . \end{aligned}$$

Therefore, the pdfs of the minimum, the maximum and the median are

$$\begin{aligned} f_{1:n}(x) &= n \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) e^{-n(\alpha x^\theta + \gamma x^\beta)} \\ &\quad \times \left(1 - \lambda + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)^{n-1} \left(1 - \lambda + 2\lambda e^{-\alpha x^\theta - \gamma x^\beta}\right) , \\ f_{n:n}(x) &= n \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) e^{-\alpha x^\theta - \gamma x^\beta} \\ &\quad \times \left[\left(1 - e^{-\alpha x^\theta - \gamma x^\beta}\right) \left(1 + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)\right]^{n-1} \left(1 - \lambda + 2\lambda e^{-\alpha x^\theta - \gamma x^\beta}\right) , \\ f_{m+1:n}(x) &= \frac{(2m+1)!}{m!m!} \left[\left(1 - e^{-\alpha x^\theta - \gamma x^\beta}\right) \left(1 + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)\right]^{m} e^{-\alpha x^\theta - \gamma x^\beta} \\ &\quad \times \left[1 - \left(1 - e^{-\alpha x^\theta} - \gamma x^\beta\right) \left(1 + \lambda e^{-\alpha x^\theta - \gamma x^\beta}\right)\right]^{m} \\ &\quad \times \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) \left(1 - \lambda + 2\lambda e^{-\alpha x^\theta - \gamma x^\beta}\right) . \end{aligned}$$

Also note that the minimum, maximum and median order statistics of the five parameters TAW distribution converges to the order statistics of several life time distributions when its parameters are changed.

6 Parameter Estimation

We now consider the maximum likelihood estimators (MLEs) of the parameters $\alpha, \beta, \theta, \gamma$, λ. Let $x_1, \dots, x_n$ be an observed random sample of size n from a $T_{AWD}(\alpha, \beta, \theta, \gamma, \lambda)$ distribution then the likelihood function can be written as

$$L(\alpha, \beta, \theta, \gamma, \lambda | x) = \prod_{i=1}^{n} \left(\alpha\theta x_i^{\theta-1} + \gamma\beta x_i^{\beta-1}\right) e^{-\alpha x_i^\theta - \gamma x_i^\beta} \left(1 - \lambda + 2\lambda e^{-\alpha x_i^\theta - \gamma x_i^\beta}\right) .$$

Hence, the log-likelihood function $\ell(\cdot) = \log L(\cdot)$ becomes

$$\begin{aligned}\ell(\alpha,\beta,\theta,\gamma,\lambda|x) &= \sum_{i=1}^{n}\log\left(\alpha\theta x_i^{\theta-1}+\gamma\beta x_i^{\beta-1}\right)-\alpha\sum_{i=1}^{n}x_i^{\theta}-\gamma\sum_{i=1}^{n}x_i^{\beta}\\ &\quad+\sum_{i=1}^{n}\log\left(1-\lambda+2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}\right). \qquad (14)\end{aligned}$$

Differentiating (14) with respect to the parameters, the components of the score vector are

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \sum_{i=1}^{n}\frac{\theta x_i^{\theta-1}}{\alpha\theta x_i^{\theta-1}+\gamma\beta x_i^{\beta-1}}-\sum_{i=1}^{n}x_i^{\theta}-\sum_{i=1}^{n}\frac{2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}x_i^{\theta}}{1-\lambda+2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}},\\ \frac{\partial\ell}{\partial\theta} &= \sum_{i=1}^{n}\frac{x_i^{\theta-1}(\theta\log(x_i)+1)}{\alpha\theta x_i^{\theta-1}+\gamma\beta x_i^{\beta-1}}-\alpha\sum_{i=1}^{n}x_i^{\theta}\log(x_i)-\sum_{i=1}^{n}\frac{2\alpha\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}x_i^{\theta}\log(x_i)}{1-\lambda+2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}},\\ \frac{\partial\ell}{\partial\gamma} &= \sum_{i=1}^{n}\frac{\beta x_i^{\beta-1}}{\alpha\theta x_i^{\theta-1}+\gamma\beta x_i^{\beta-1}}-\sum_{i=1}^{n}x_i^{\beta}-\sum_{i=1}^{n}\frac{2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}x_i^{\beta}}{1-\lambda+2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}},\\ \frac{\partial\ell}{\partial\beta} &= \sum_{i=1}^{n}\frac{x_i^{\beta-1}(\beta\log(x_i)+1)}{\alpha\theta x_i^{\theta-1}+\gamma\beta x_i^{\beta-1}}-\gamma\sum_{i=1}^{n}x_i^{\beta}\log(x_i)-\sum_{i=1}^{n}\frac{2\lambda\gamma e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}x_i^{\beta}\log(x_i)}{1-\lambda+2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}},\\ \frac{\partial\ell}{\partial\lambda} &= \sum_{i=1}^{n}\frac{2e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}-1}{1-\lambda+2\lambda e^{-\alpha x_i^{\theta}-\gamma x_i^{\beta}}}.\end{aligned}$$

The maximum likelihood estimators $\hat{\alpha},\hat{\theta},\hat{\gamma},\hat{\beta},\hat{\lambda}$ of $\alpha,\theta,\gamma,\beta,\lambda$ are obtained by setting the score vector to zero and solving the system of nonlinear equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the log-likelihood function given in (14). For the five parameters TAW distribution all second order derivatives exist. Thus, we have

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\theta}\\ \hat{\beta}\\ \hat{\gamma}\\ \hat{\lambda}\end{pmatrix} \sim \text{Normal}\left(\begin{pmatrix}\alpha\\ \theta\\ \beta\\ \gamma\\ \lambda\end{pmatrix},\Sigma\right)$$

with

$$\Sigma = -\mathrm{E}\begin{bmatrix}V_{\alpha\alpha} & V_{\alpha\theta} & V_{\alpha\beta} & V_{\alpha\gamma} & V_{\alpha\lambda}\\ V_{\theta\alpha} & V_{\theta\theta} & V_{\theta\beta} & V_{\theta\gamma} & V_{\theta\lambda}\\ V_{\beta\alpha} & V_{\beta\theta} & V_{\beta\beta} & V_{\beta\gamma} & V_{\beta\lambda}\\ V_{\gamma\alpha} & V_{\gamma\theta} & V_{\gamma\beta} & V_{\gamma\gamma} & V_{\gamma\lambda}\\ V_{\lambda\alpha} & V_{\lambda\theta} & V_{\lambda\beta} & V_{\lambda\gamma} & V_{\lambda\lambda}\end{bmatrix}^{-1}.$$

Here $V_{..}$ denotes the second derivative of the log-likelihood function with respect to the two parameters in the index.

By calculating this inverse matrix this will yield asymptotic variance and covariances of the MLEs. Approximate $100(1-\phi)\%$ confidence intervals for the parameters can be determined as

$$\hat{\alpha} \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\alpha\alpha}}\,, \quad \hat{\theta} \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\theta\theta}}\,, \quad \hat{\beta} \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\beta\beta}}\,, \quad \hat{\gamma} \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\gamma\gamma}}\,, \quad \hat{\lambda} \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\lambda\lambda}}\,,$$

where z_ϕ is the upper ϕth percentile of the standard normal distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihood functions to obtain likelihood ratio test statistics for testing the sub-model of the new distribution. For example, we can then use the test statistic to check whether the TAW distribution is statistically "superior" to an AW distribution for a given data set. In this case we can compare the first model against the second model by testing $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$.

7 Application

We now provide a little data analysis in order to assess the goodness-of-fit of the model. The data refers to the lifetimes of $n = 50$ devices provided by Aarset (1987) and is often cited as an example with bathtub shaped failure rate. Table 3 gives a descriptive summary of the data.

Table 3: Descriptive statistics of the lifetime data

n	Mean	Median	Variance	Minimum	Maximum
50	45.69	48.50	1078.15	0.1	86.0

We have used the Weibull, the AW and the TAW distribution to analyze the data. In addition to the MLEs, also the Akaike Information Criterion (AIC) by Akaike (1974), which is used to select the best model among several models, is provided in Table 4. The AIC is given as $AIC = -2\ell + 2p$, where p is the number of parameters in the model. The result indicates that a TAW distribution has the lowest AIC so it fits best to this data. The statistical software R (Ihaka and Gentleman, 1996) has been used to perform all necessary calculations and to produce all graphics.

8 Concluding Remarks

In the present study, we have introduced a new generalization of the AW distribution called the TAW distribution. The 4-parameter AW distribution is embedded in the proposed distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function and reliability behavior of the TAW distribution shows that the subject distribution can be used to model reliability data. We have presented an example where a TAW distribution fits better than the AW distribution. We believe that the subject distribution can be used in several different areas. We expect that this study will serve as a reference and help to advance future research in the subject area.

Table 4: MLEs under the considered models and corresponding AIC values

Model	MLEs	$\ell(\cdot\|x)$	AIC
$W(\gamma,\beta)$	$\hat{\gamma}=0.0270$, $\hat{\beta}=0.9490$	-241.0018	486.0036
$AW(\alpha,\theta,\gamma,\beta)$	$\hat{\alpha}=0.0004$, $\hat{\theta}=1.8527$	-231.9367	471.8734
	$\hat{\gamma}=0.1118$, $\hat{\beta}=0.2889$		
$TAW(\alpha,\theta,\gamma,\beta,\lambda)$	$\hat{\alpha}=0.0001$, $\hat{\theta}=2.1463$	-229.4571	468.9142
	$\hat{\gamma}=0.0834$, $\hat{\beta}=0.4152$		
	$\hat{\lambda}=0.0077$		

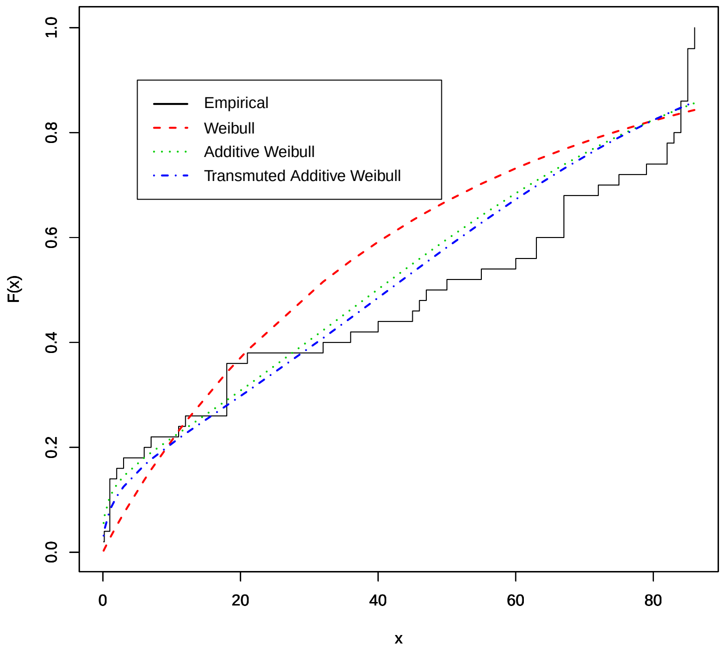

Figure 4: Empirical cdf compared with Weibull, additive Weibull, and transmuted additive Weibull fits of the life time data

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