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Fuzzy Probability Spaces and Their Applications in Decision Making

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Abstract: In this paper, two types of fuzzy probability spaces will be introduced and their applications in methods of decision making under risk (especially in the Decision Matrix Method) will be described. First, a fuzzy probability space that generalizes the classical probability space $(\Re^n, \mathcal{B}_n, p)$ to the situation of fuzzy random events will be studied. It will be applied to perform fuzzy discretization of continuous risk factors. Second, a fuzzy probability space that enables an adequate mathematical modelling of expertly set uncertain probabilities of states of the world will be defined. The presented theoretical results will be illustrated with two examples comparing stock yields.

Keywords: Decision Making under Risk, Fuzzy Events, Fuzzy Probabilities, Fuzzy Decision Matrix.

1 Introduction

In decision making under risk, decision matrices are often used as a tool of risk analysis. They describe how consequences of alternatives depend on the fact which of possible and mutually disjoint states of the world occurs. The states of the world are supposed to be set exactly and their probabilities to be known. However in practice, the states of the world are often specified only vaguely and their probabilities are based on experts' estimations. Sometimes the states of the world and their probability distributions are obtained as a result of discretization of continuous risk factors. In this paper, it will be shown how the apparatus of fuzzy sets, especially two types of fuzzy probability spaces, can be used in these situations.

In general, various ways of fuzzification of probability spaces are described in Dubois and Prade (1980). Many bibliographical references concerning mathematical models combining fuzzy and stochastic approaches are available in Dubois et al. (2000).

2 Applied Notions of the Fuzzy Sets Theory

A fuzzy set A on a universal set X, $X \neq \emptyset$, is uniquely determined by its membership function $A : X \to [0,1]$. A set $SuppA = \{x \in X | A(x) > 0\}$ is called a support of A. Sets $A_\alpha = \{x \in X | A(x) \geq \alpha\}$, $\alpha \in (0,1]$, are called α-cuts of A. A set $KerA = \{x \in X | A(x) = 1\}$ is a kernel of A. A fuzzy set A is called normal if $KerA \neq \emptyset$. The family of all fuzzy sets on X is denoted by $\mathcal{F}(X)$.

A normal fuzzy set A on the set of all real numbers $\Re$, whose α-cuts A_α, $\alpha \in (0,1]$, are closed intervals, and whose support $SuppA$ is bounded, is called a fuzzy number.

The family of all fuzzy numbers is denoted by $\mathcal{F}_{\mathcal{N}}(\Re)$. A fuzzy number A is said to be defined on $[a,b]$ if $Supp A \subseteq [a,b]$. The family of all fuzzy numbers on $[a,b]$ is denoted by $\mathcal{F}_{\mathcal{N}}([a,b])$. In this paper, trapezoidal fuzzy numbers will be used. The membership function of any trapezoidal fuzzy number A is piecewise linear, determined by four points $(a^1,0), (a^2,1), (a^3,1), (a^4,0)$, $a^1 \le a^2 \le a^3 \le a^4$. Real numbers a^1, a^2, a^3, a^4 are called significant values of A.

Calculations with fuzzy numbers are based on a so-called extension principle. For example, let $*$ be a binary operation on $\Re$. Its extension to $\mathcal{F}_{\mathcal{N}}(\Re)$ is defined for any $A, B \in \mathcal{F}_{\mathcal{N}}(\Re)$ and any $z \in \Re$ as follows

$$(A * B)(z) = \begin{cases} \sup\{\min\{A(x), B(y)\} \mid z = x * y,\ x, y \in \Re\} & \text{if such } x, y \text{ exist,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The center of gravity of a fuzzy number A defined on $[a,b]$ is a real number $c_A \in [a,b]$ given by the following formula

$$c_A = \frac{\int_a^b A(x) x \, dx}{\int_a^b A(x) \, dx} \,. \tag{2}$$

The linguistic approximation of a fuzzy number A on $[a,b]$ by means of a set of linguistic terms $\{\mathcal{T}_1, \ldots, \mathcal{T}_s\}$, where meanings of $\mathcal{T}_k$ are fuzzy numbers T_k on $[a,b]$ for $k = 1, \ldots, s$, is the linguistic term $\mathcal{T}_{k_0}$, $k_0 \in \{1, \ldots, s\}$, such that

$$P(A, T_{k_0}) = \max_{k=1,\ldots,s} P(A, T_k) \,, \tag{3}$$

where

$$P(A, T_k) = 1 - \frac{\int_a^b |A(x) - T_k(x)| \, dx}{\int_a^b (A(x) + T_k(x)) \, dx} \,, \qquad k = 1, \ldots, s \,. \tag{4}$$

A fuzzy scale on $[a,b]$ is a finite set of fuzzy numbers $A_1, \ldots, A_n$ that are defined on $[a,b]$ and form a fuzzy partition on $[a,b]$, i.e. $\sum_{i=1}^n A_i(x) = 1$ holds for any $x \in [a,b]$. If a fuzzy scale expresses a mathematical meaning of a natural linguistic scale, then it is called a linguistic fuzzy scale. Fuzzy scales enable finite fuzzy representations of intervals.

Fuzzy numbers V_i, $i = 1, \ldots, m$, defined on $[0,1]$ are called normalized fuzzy weights if for all $\alpha \in (0,1]$ and for all $i \in \{1, \ldots, m\}$ the following holds: for any $v_i \in V_{i\alpha}$ there exist $v_j \in V_{j\alpha}$, $j = 1, \ldots, m$, $j \ne i$, such that

$$v_i + \sum_{j=1, j \ne i}^{m} v_j = 1 \,. \tag{5}$$

Normalized fuzzy weights express uncertain rates, a division of a unit into uncertain parts.

The fuzzy weighted average of fuzzy numbers U_i, $i = 1, \ldots, m$, defined on $[a,b]$, with normalized fuzzy weights V_i, is a fuzzy number U on $[a,b]$ whose membership function is for any $u \in [a,b]$ given by the formula

$$\begin{aligned} U(u) = \max\{ \ & \min\{V_1(v_1), \ldots, V_m(v_m), U_1(u_1), \ldots, U_m(u_m)\} | \\ & \textstyle\sum_{i=1}^m v_i u_i = u \,, \sum_{i=1}^m v_i = 1 \,, v_i \ge 0 \,, u_i \in [a,b] \,, i = 1, \ldots, m\} \,. \end{aligned} \tag{6}$$

The following notation will be used for the fuzzy weighted average:

$$U = (\mathcal{F}) \sum_{i=1}^{m} V_i U_i \,. \tag{7}$$

3 Fuzzy Discretization of Continuous Risk Factors

A decision matrix represents a suitable instrument of risk analysis when risk factors affecting the consequences of alternatives are of discrete nature, and at the same time each of them takes on a relatively small number of values. Then states of the world are given by all possible combinations of values of the risk factors.

In practice, the Decision Matrix Method is applied also in situations when some of the risk factors are continuous. Then their continuous probability distributions have to be approximated by discrete ones. The approximation is more realistic when continuous domains (intervals) of the risk factors are replaced by fuzzy scales instead of usual crisp scales.

For that purpose, it is necessary to generalize the classical probability space $(\Re^n, \mathcal{B}_n, p)$, where $\Re$ is the set of real numbers, $\mathcal{B}_n$ is the σ-field of all Borel subsets on $\Re^n$, and p is a probability measure defined on $\mathcal{B}_n$, to the situation of fuzzy events. The associated fuzzy probability space will be defined as a triple $(\Re^n, \mathcal{F}_\mathcal{B}(\Re^n), P)$, where $\mathcal{F}_\mathcal{B}(\Re^n)$ is the family of all the fuzzy sets on $\Re^n$ whose membership functions are Borel measurable, and the probability $P(A)$ of any fuzzy event $A \in \mathcal{F}_\mathcal{B}(\Re^n)$ is given by the following formula

$$P(A) = \int_{\Re^n} A(x)\, dp(x)\,. \tag{8}$$

It can be proved (see Negoita and Ralescu, 1975) that $\mathcal{F}_\mathcal{B}(\Re^n)$ has the following properties: a) $\Re^n \in \mathcal{F}_\mathcal{B}(\Re^n)$, b) if $A \in \mathcal{F}_\mathcal{B}(\Re^n)$, then $\overline{A} \in \mathcal{F}_\mathcal{B}(\Re^n)$, c) if $A_i \in \mathcal{F}_\mathcal{B}(\Re^n)$, $i = 1, 2, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_\mathcal{B}(\Re^n)$; It means $\mathcal{F}_\mathcal{B}(\Re^n)$ is a σ-field of fuzzy sets on $\Re^n$. Moreover, the mapping $P : \mathcal{F}_\mathcal{B}(\Re^n) \to [0, 1]$ given by (8) satisfies the classical axioms of probability (see Negoita and Ralescu, 1975). Therefore, it is meaningful to call the triple $(\Re^n, \mathcal{F}_\mathcal{B}(\Re^n), P)$ a fuzzy probability space. And evidently, if a set A belongs to $\mathcal{B}_n$, then it belongs also to $\mathcal{F}_\mathcal{B}(\Re^n)$ as a fuzzy set, and $P(A) = p(A)$. So, $(\Re^n, \mathcal{F}_\mathcal{B}(\Re^n), P)$ is an extension of $(\Re^n, \mathcal{B}_n, p)$.

Let us notice that the probability of a fuzzy event $A \in \mathcal{F}_\mathcal{B}(\Re^n)$ could be expressed also in another way – as a fuzzy set $P_F(A)$ on $[0, 1]$. Its membership function would be defined for any $\tilde{p} \in [0, 1]$ by the following formula

$$P_F(A)(\tilde{p}) = \begin{cases} \sup\{\alpha \in (0,1] \mid \tilde{p} = p(A_\alpha)\} & \text{if } \{\alpha \in (0,1] \mid \tilde{p} = p(A_\alpha)\} \neq \emptyset\,, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

It means, the fuzzy probability $P_F(A)$ is uniquely determined by the probabilities of α-cuts of A, $p(A_\alpha)$, $\alpha \in (0, 1]$. The following relation between $P(A)$, introduced by (8), and $p(A_\alpha)$, $\alpha \in (0, 1]$, determining $P_F(A)$, holds for any fuzzy event $A \in \mathcal{F}_\mathcal{B}(\Re^n)$

$$P(A) = \int_0^1 p(A_\alpha)\, d\alpha\,. \tag{10}$$

Let us prove the above proposition. Interpreting A_α, $\alpha \in (0, 1]$, as fuzzy sets, we can write $\int_0^1 p(A_\alpha)\, d\alpha = \int_0^1 \int_{\Re^n} A_\alpha(x)\, dp(x)\, d\alpha$. By the Fubini theorem, $\int_0^1 \int_{\Re^n} A_\alpha(x)\, dp(x)\, d\alpha = \int_{\Re^n} \int_0^1 A_\alpha(x)\, d\alpha\, dp(x)$. Since $\int_0^1 A_\alpha(x)\, d\alpha = A(x)$ for any $x \in \Re$, $\int_{\Re^n} \int_0^1 A_\alpha(x)\, d\alpha\, dp(x)$ $= \int_{\Re^n} A(x)\, dp(x) = P(A)$, which completes the proof.

As the fuzzy probability P_F seems to be too complicated to be used in practice, the crisp probability P will be preferred in this paper.

Now, it will be shown how the fuzzy probability space can be applied to perform fuzzy discretization of continuous risk factors in decision making under risk.

First, let us suppose that consequences of alternatives are affected by only one continuous risk factor Z whose probability distribution is given by a density function $f(z)$. Consider a fuzzy scale $A_1, \ldots, A_n$ on the domain of the risk factor. As elements of the fuzzy scale are fuzzy random events, their probabilities $P(A_i)$, $i = 1, \ldots, n$, are given by (8), i.e.

$$P(A_i) = \int_{Supp A_i} A_i(z) f(z)\, dz\,. \tag{11}$$

It is easy to check that $\sum_{i=1}^{n} P(A_i) = 1$ and $P(A_i) \geq 0$, $i = 1, \ldots, n$. So, a discrete probability distribution is defined on the given fuzzy scale.

If the density function of the risk factor Z is not known, a similar probability distribution on the given fuzzy scale can be derived directly from measured data. If measurements $z_1, \ldots, z_m$ of Z are given, $m \gg n$, then probabilities of the fuzzy scale elements can be set by the formula

$$P(A_i) = \frac{1}{m} \sum_{j=1}^{m} A_i(z_j)\,, \qquad i = 1, \ldots, n\,. \tag{12}$$

The fuzzy expected value and the fuzzy standard deviation of such a fuzzy random variable Z that takes on values A_i of the given fuzzy scale with probabilities $P(A_i)$, $i = 1, \ldots, n$, are defined by the following formulas

$$FEZ = \sum_{i=1}^{n} P(A_i) A_i\,, \qquad F\sigma Z = \sqrt{\sum_{i=1}^{n} P(A_i)(A_i - FEZ)^2}\,. \tag{13}$$

Now let us suppose that consequences of alternatives are affected by several independent continuous risk factors. Then the above described procedure of fuzzy discretization is applied to each of them. All combinations of fuzzy values of the risk factors determine the states of the world; probabilities of the states are given as products of probabilities of particular fuzzy values of the risk factors.

If the consequences of alternatives are affected for example by two mutually dependent continuous risk factors Y, Z, whose conjugate probability distribution is given by a density function $f(y, z)$, then fuzzy scales $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ will be defined on domains of both risk factors. The Cartesian products $A_i \times B_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where $(A_i \times B_j)(y, z) = A_i(y) B_j(z)$ for any $y, z \in \Re$, form a fuzzy partition on the Cartesian product of domains of both risk factors. A conjugate discrete probability distribution of the fuzzy discretized risk factors is given by the following probability function

$$P(A_i, B_j) = \int_{Supp(A_i \times B_j)} (A_i \times B_j)(y, z) f(y, z)\, dy\, dz\,, \tag{14}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, m$. Possible states of the world are determined by such combinations of fuzzy values of the risk factors for which $P(A_i, B_j) > 0$.

Example 1. The fuzzy discretization procedure will be illustrated with an example concerning stock yields. Let us suppose that yields (in %) on stocks A and B are continuous random variables with normal probability distributions whose parameters are $\mu_A = 4$, $\sigma_A = 7.5$, and $\mu_B = 6.5$, $\sigma_B = 12$, respectively.

To discretize the real variable "stock yield", a linguistic fuzzy scale {NH, NVL, NL, NM, NS, AZ, PS, PM, PL, PVL, PH} (see Figure 1 and Table 1) will be defined on the closed interval $[-50, 50]$ of possible stock yields.

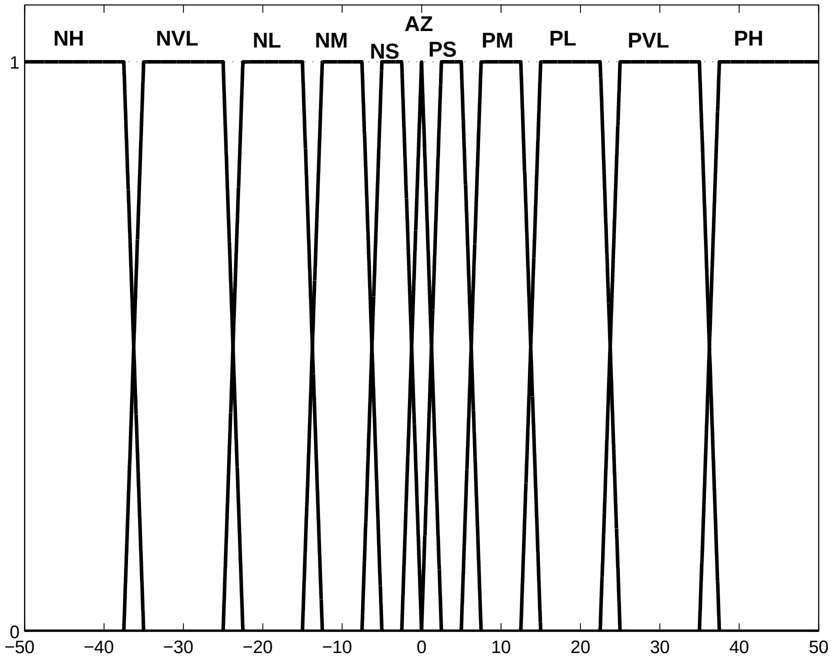

Figure 1: Linguistic fuzzy scale for the variable "stock yield"

Table 1: Discrete probability distributions of fuzzy stock yields of A and B

					Probability	
Linguistic description	Fuzzy Stock Yield (%)				A	B
Negative Very Large (NVL)	−37.5	−35	−25	−22.5	0	0.01
Negative Large (NL)	−25	−22.5	−15	−12.5	0.01	0.03
Negative Medium (NM)	−15	−12.5	−7.5	−5	0.08	0.11
Negative Small (NS)	−7.5	−5	−2.5	0	0.16	0.11
Approximately Zero (AZ)	−2.5	0	0	2.5	0.11	0.07
Positive Small (PS)	0	2.5	5	7.5	0.26	0.16
Positive Medium (PM)	5	7.5	12.5	15	0.29	0.27
Positive Large (PL)	12.5	15	22.5	25	0.09	0.17
Positive Very Large (PVL)	22.5	25	35	37.5	0	0.07

Discrete probability distributions on the given linguistic fuzzy scale are calculated for the yields on stocks A and B by means of the formula (11); they are displayed in Table 1. The fuzzy expected values and fuzzy standard deviations of the fuzzy stock yields are calculated according to (13); significant values of the fuzzy numbers are displayed in Table 2.

Table 2: Fuzzy expected values and fuzzy standard deviations

	Fuzzy Stock Yield (%)			
FEA	−0.35	2.15	5.80	8.30
FEB	1.28	3.78	8.65	11.5
$F\sigma A$	2.88	4.99	11.33	15.85
$F\sigma B$	5.09	8.02	16.66	21.16

Table 3: Linguistic approximation of results

Fuzzy number	Linguistic Approximation
FEA	PS
FEB	PS or PM
$[FEA - F\sigma A, FEA + F\sigma A]$	NS or AZ or PS or PM
$[FEB - F\sigma B, FEB + F\sigma B]$	NS or AZ or PS or PM or PL

The linguistic approximation defined by (3) and (4) makes it possible to characterize both the fuzzy random variables also linguistically (Table 3). Judging by the results of linguistic approximation, we can say that the stock B is better than the stock A with respect to the criterion of yield.

4 Expertly Defined Fuzzy Probabilities of States of the World

Not in all cases, when the Decision Matrix Method is applied, probabilities of states of the world represent the results of exhaustive mathematical risk analysis. Especially if states of the world are specified linguistically or if they are given by a large number of hardly describable risk factors, their probabilities are set only on the basis of experts' knowledge and experience.

To enable a correct mathematical modelling of uncertain probabilities of states of the world, it is necessary to extend the classical probability space $(\Omega, \mathcal{P}(\Omega), p)$, where $\Omega = \{\omega_1, \dots, \omega_r\}$, $p(\omega_i) = p_i$, $p_i > 0$, $i = 1, \dots, r$, $\sum_{i=1}^r p_i = 1$, $\mathcal{P}(\Omega)$ is the family of all subsets of Ω, and $p(A) = \sum_{\{i:\omega_i \in A\}} p_i$ for any $A \in \mathcal{P}(\Omega)$, to the situation of fuzzy probabilities of elementary events (fuzzy random events will be allowed as well).

Let us define the corresponding fuzzy probability space with a finite set of elementary events as a triple $(\Omega, \mathcal{F}(\Omega), P)$, where $\Omega = \{\omega_1, \dots, \omega_r\}$ is a set of elementary events whose fuzzy probabilities are given by normalized fuzzy weights [1] $P_1, \dots, P_r$, $P_i \neq 0$ for $i = 1, \dots, r$, $\mathcal{F}(\Omega)$ is the family of all fuzzy sets on Ω (fuzzy random events), and a mapping $P : \mathcal{F}(\Omega) \to \mathcal{F}_{\mathcal{N}}([0, 1])$ assigns to each fuzzy random event $A \in \mathcal{F}(\Omega)$ its fuzzy probability $P(A)$ according to the formula

$$P(A) = (\mathcal{F}) \sum_{i=1}^{r} P_i A(\omega_i)\,, \tag{15}$$

[1] Modelling fuzzy probabilities of elementary events, the authors of this paper apply a general structure of fuzzy numbers, called normalized fuzzy weights, originally developed in Pavlačka (2004) for the purpose of aggregation of partial evaluations in MCDM. The same structure of fuzzy probabilities is also used in Pan and Yuan (1997) but there the authors take as their starting point interval probabilities (see Pan and Klir, 1997, Campos, Huete, and Moral, 1994)

where $A(\cdot)$ is the membership function of A.

The mapping P has properties representing a generalization of axioms of the classical probability: a) $P(\Omega) = 1$, because $P(\Omega) = (\mathcal{F})\sum_{i=1}^{r} P_i \Omega(\omega_i) = (\mathcal{F})\sum_{i=1}^{r} P_i \cdot 1$, and by the definition of fuzzy weighted average, $(\mathcal{F})\sum_{i=1}^{r} P_i \cdot 1 = 1$. b) It is evident that $P(A) \geq 0$ holds for any $A \in \mathcal{F}(\Omega)$. c) Finally, for any $A_1, \dots, A_s \in \mathcal{F}(\Omega)$, such that $A_k \cap A_l = \emptyset$ for $k \neq l$, the following holds

$$P(\cup_{j=1}^{s} A_j) = (\mathcal{F})(P(A_1) \cdot 1 + \dots + P(A_s) \cdot 1 + P(\overline{\cup_{j=1}^{s} A_j}) \cdot 0)\,. \tag{16}$$

Let us prove c) in detail. As by assumption $(A_k \cap A_l)(\omega_i) = \min\{A_k(\omega_i), A_l(\omega_i)\} = 0$ for any $k, l \in \{1, \dots, s\}$, $k \neq l$, and any $i \in \{1, \dots, r\}$, we can write $(\cup_{j=1}^{s} A_j)(\omega_i) = \max_{j=1,\dots,s}\{A_j(\omega_i)\} = \sum_{j=1}^{s} A_j(\omega_i)$. Consider $p_i \in P_{i\alpha}$, $i = 1, \dots, r$, $\sum_{i=1}^{r} p_i = 1$. It holds $\sum_{i=1}^{r} p_i(\cup_{j=1}^{s} A_j)(\omega_i) = \sum_{i=1}^{r} p_i \sum_{j=1}^{s} A_j(\omega_i) = \sum_{j=1}^{s}\sum_{i=1}^{r} p_i A_j(\omega_i) = \sum_{j=1}^{s} p_j^*$. Real numbers $p_j^* = \sum_{i=1}^{r} p_i A_j(\omega_i)$ are evidently elements of $P(A_j)_\alpha$ for $j = 1, \dots, s$. Let us denote $A_{s+1} = \overline{\cup_{j=1}^{s} A_j}$ and $p_{s+1}^* = \sum_{i=1}^{r} p_i A_{s+1}(\omega_i)$; then similarly $p_{s+1}^* \in P(A_{s+1})_\alpha$. Since $A_1, \dots, A_s, A_{s+1}$ form a fuzzy partition of Ω, $\sum_{j=1}^{s+1} p_j^* = \sum_{i=1}^{r} p_i \sum_{j=1}^{s+1} A_j(\omega_i) = \sum_{i=1}^{r} p_i = 1$. By the above, the following holds for any $p \in [0,1]$:

$$\begin{aligned} P(\bigcup_{j=1}^{s} A_j)(p) &= \max\{\min_{1\leq i\leq r}\{P_i(p_i)\} \mid p = \sum_{i=1}^{r} p_i(\cup_{j=1}^{s} A_j)(\omega_i),\ \sum_{i=1}^{r} p_i = 1,\ p_i \in [0,1]\} \\ &= \max\{\min_{1\leq j\leq s+1}\{P(A_j)(p_j^*)\} \mid p = \sum_{j=1}^{s} p_j^*,\ \sum_{j=1}^{s+1} p_j^* = 1,\ p_j^* \in [0,1]\} \\ &= ((\mathcal{F})P(A_1) \cdot 1 + \dots + P(A_s) \cdot 1 + P(\overline{\cup_{j=1}^{s} A_j}) \cdot 0))(p)\,. \end{aligned}$$

Similarly, it can be easily proved, that the family of all fuzzy sets on Ω forms a σ-field of fuzzy sets. So, it is clearly meaningful to call the triple $(\Omega, \mathcal{F}(\Omega), P)$ a fuzzy probability space.

In a fuzzy probability space $(\Omega, \mathcal{F}(\Omega), P)$ with a finite set Ω, any mapping $U : \Omega \to \mathcal{F}_{\mathcal{N}}(\Re)$ defines a discrete fuzzy random variable. For example, fuzzy evaluations of an alternative under possible states of the world whose fuzzy probabilities are known represent a discrete fuzzy random variable. The probability distribution of a discrete fuzzy random variable U is given by a mapping $P(U_i) = P_i$, $i = 1, \dots, r$, where $U_i = U(\omega_i)$ and $P_i = P(\omega_i)$.

The expected fuzzy value FEU of a discrete fuzzy random variable U is defined as the fuzzy weighted average of fuzzy values $U_1, \dots, U_r$ with normalized fuzzy weights $P_1, \dots, P_r$, i.e.

$$FEU = (\mathcal{F})\sum_{i=1}^{r} P_i U_i\,. \tag{17}$$

5 Fuzzy Decision Matrices

Let us consider a problem of decision making under risk that is described by the following fuzzy decision matrix, where $x_1, \dots, x_n$ are alternatives, $S_1, \dots, S_r$ states of the world, $P_1, \dots, P_r$ their fuzzy probabilities, and $U_{i,k}$, $i = 1, \dots, n$, $k = 1, \dots, r$, denote fuzzy degrees in which the alternatives x_i satisfy a given decision objective if the states S_k occur.

Table 4: Fuzzy decision matrix

	S_1	S_2	$\dots$	S_r	
	P_1	P_2	$\dots$	P_r	FEU
x_1	$U_{1,1}$	$U_{1,2}$	$\dots$	$U_{1,r}$	FEU_1
x_2	$U_{2,1}$	$U_{2,2}$	$\dots$	$U_{2,r}$	FEU_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$U_{n,1}$	$U_{n,2}$	$\dots$	$U_{n,r}$	FEU_n

Fuzzy numbers FEU_i express expected fuzzy evaluations of alternatives x_i for $i = 1, \dots, n$; it means they are calculated according to the formula

$$FEU_i = (\mathcal{F}) \sum_{k=1}^{r} P_k U_{i,k} \, . \tag{18}$$

The best alternative will be chosen by the rule of the maximum expected fuzzy evaluation. For that purpose some of the preference relations on $\mathcal{F}_{\mathcal{N}}(\Re)$ (see Talašová, 2003), for example the ordering of fuzzy numbers according to their centers of gravity, can be applied; also the linguistic approximation of the expected fuzzy evaluations by means of an ordered set of evaluating linguistic terms can be used.

A similar approach can be applied also to multicriteria decision making under risk. In Talašová (2005), utilization of expertly defined fuzzy probabilities in three-dimensional decision matrices is presented, where multicriteria evaluating procedures are based either on fuzzy weighted averages of partial fuzzy evaluations or on fuzzy expert systems (see also Talašová, 2000, Talašová, 2003).

Example 2. Let us consider a problem of comparing two stocks, C and D, with respect to their future yields. The starting prices of C and D are 2900 and 3300 CZK, respectively. The considered states of the world are *economic drop*, *economic stagnation* and *economic growth*; their uncertain probabilities will be set expertly. As fuzzy probabilities have to form the structure of normalized fuzzy weights, it is not so easy for an expert to define them directly. So, the expert expresses his/her estimates of the probabilities by fuzzy numbers P'_1, P'_2, P'_3 whose significant values have to satisfy at least the following natural conditions

$$\sum_{i=1}^{3} p_i'^1 \le 1 \, , \quad \sum_{i=1}^{3} p_i'^2 \le 1 \, , \quad \sum_{i=1}^{3} p_i'^3 \ge 1 \, , \quad \sum_{i=1}^{3} p_i'^4 \ge 1 \, . \tag{19}$$

The correct fuzzy probabilities P_1, P_2, P_3 are obtained from P'_1, P'_2, P'_3 by the transformation

$$\begin{aligned} p_i^1 &= \max\{p_i'^1, 1 - \textstyle\sum_{j=1, j\neq i}^{3} p_j'^4\} \, , \quad & p_i^3 &= \min\{p_i'^3, 1 - \textstyle\sum_{j=1, j\neq i}^{3} p_j'^2\} \, , \\ p_i^2 &= \max\{p_i'^2, 1 - \textstyle\sum_{j=1, j\neq i}^{3} p_j'^3\} \, , \quad & p_i^4 &= \min\{p_i'^4, 1 - \textstyle\sum_{j=1, j\neq i}^{3} p_j'^1\} \, , \end{aligned} \tag{20}$$

where $i = 1, 2, 3$. The transformation, that was described for interval probabilities in Campos et al. (1994), eliminates the inconsistency of experts' estimates (see Figure 2).

The future stock prices of C and D under the given states of the economy are also estimated by fuzzy numbers. Corresponding fuzzy stock yields (in %) are calculated by means of the extension principle according to the formula

$$\text{yield} = \frac{\text{new price} - \text{old price}}{\text{old price}} \cdot 100 \, . \tag{21}$$

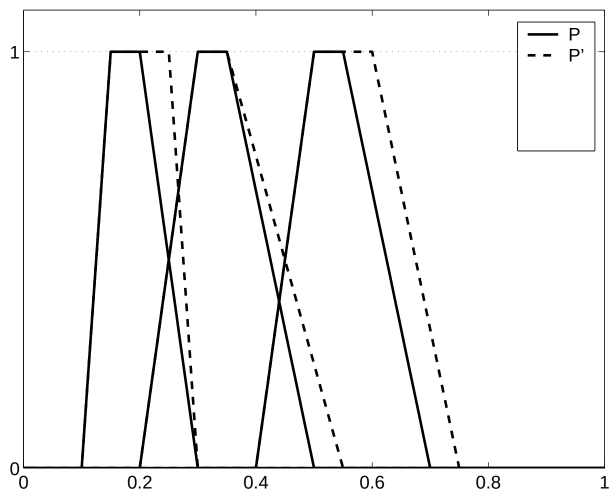

Figure 2: Fuzzy probabilities of states of the economy

Significant values of all the fuzzy numbers are given in Table 5. The expected fuzzy

Table 5: Fuzzy decision matrix for stocks C and D

	Economic Drop				Econ. Stagnation				Economic Growth			
P	0.1	0.15	0.2	0.3	0.4	0.5	0.55	0.7	0.2	0.3	0.35	0.5
C-price	2200	2400	2600	2800	2750	2900	3000	3150	3100	3250	3400	3600
C-yield	−24.1	−17.2	−10.3	−3.5	−5.2	0	3.5	8.6	6.9	12.1	17.2	24.1
D-price	2400	2550	2750	2900	3000	3150	3500	3700	3750	3900	4100	4300
D-yield	−27.3	−22.7	−16.7	−12.1	−9.1	−4.6	6.1	12.1	13.6	18.2	24.2	30.3

Table 6: Expected fuzzy stock yields, their centers of gravity

	Exp. Fuzzy Stock Yield				Center of Gravity
FEC	−8.45	0.17	6.21	15.17	3.29
FED	−10	−1.37	9.01	18.79	4.15

stock yields presented in Table 6 were calculated by the formula (18). The fuzzy numbers FEC, FED are incomparable in the sense of the usual ordering of fuzzy numbers that is based on the ordering of their α-cuts. But as the centre of gravity of FED is greater than that of FEC, the stock D seems to be better than C with respect to the criterion of yield.

6 Conclusion

Mathematical models of decision making under risk can be more realistic if the apparatus of the fuzzy sets theory is applied. Fuzzy discretization of continuous risk factors is more natural than the crisp one. Similarly, vague expert's information concerning probabilities of states of the world is expressed in a more adequate way if fuzzy probabilities are used.

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