AUSTRIAN JOURNAL OF STATISTICS
Volume 35 (2006), Number 2&3, 307–313

Modelling of Economic Time Series and the Method of Cointegration

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Abstract: The article is focused on the problem of modelling multidimensional non-stationary cointegrated processes. It is a modern method especially used for the description of multidimensional economic time series. The multidimensional process Y_t is called cointegrated with the cointegrating vector β, if the process $\beta' Y_t$ is stationary or trend-stationary. For instance this property can be found in some series of economic indices which are predominantly non-stationary. Methods connected with estimates of cointegrating vectors and with a cointegration testing are applied to economic data. All the methods given were programmed in the computing environment MATLAB.

Keywords: Unit Root Tests, Tests of Cointegration, Exchange Rates.

1 Introduction

Cointegration is a modern method of modelling multidimensional non-staionary time series, especially economic time series. Detailed descriptions of this method are found for example in Engle and Granger (1987), Hamilton (1994), Johansen (1995), or Banerjee et al. (1993).

This article deals with the analysis of some exchange rates in terms of cointegration. Two methods of estimation of a cointegrating relation (vector) are mentioned. The first method is based on linear regression, the second on maximum likelihood. The notation, definitions, and procedures described in Hamilton (1994), Johansen (1995) are used here.

2 Basic Definitions

Definition 1 Let $\{\boldsymbol{\epsilon}_t\}$ be a sequence of independent identically distributed random variables ($i.i.d.$) with zero mean and variance matrix $\boldsymbol{\Omega}$. A stochastic process $\{\mathbf{Y}_t\}$ which satisfies that $\mathbf{Y}_t - \mathrm{E}\mathbf{Y}_t = \sum_{i=1}^{\infty} \mathbf{C}_i \boldsymbol{\epsilon}_{t-i}$ is called $I(0)$ process if $\mathbf{C} = \sum_{i=0}^{\infty} \mathbf{C}_i \neq \mathbf{0}$.

Definition 2 A stochastic process $\{\mathbf{Y}_t\}$ is called *integrated of order* d, $I(d)$, $d = 1, 2, \ldots$, if $\Delta^d(\mathbf{Y}_t - \mathrm{E}\mathbf{Y}_t)$ is a $I(0)$ process (Δ denotes the difference operator).

We will focus on $I(0)$ and $I(1)$ processes in this paper. Let in the following $\boldsymbol{\epsilon}_t$ be a sequence of independent normally distributed n-dimensional variables $\boldsymbol{\epsilon}_t \sim N_n(\mathbf{0}, \boldsymbol{\Omega})$.

Definition 3 A stochastic process $\{\mathbf{Y}_t\}$ is called n-dimensional autoregressive process VAR(p), if

$$\mathbf{Y}_t = \boldsymbol{\Phi}_1 \mathbf{Y}_{t-1} + \cdots + \boldsymbol{\Phi}_p \mathbf{Y}_{t-p} + \boldsymbol{\Lambda} \mathbf{D}_t + \boldsymbol{\epsilon}_t \,, \quad \text{for } t = 1, \ldots, T \tag{1}$$

for fixed values of $\mathbf{Y}_{-p+1}, \ldots, \mathbf{Y}_0$, where $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_p$ are matrices of coefficients $(n \times n)$, $\mathbf{\Lambda}$ is a $n \times s$ matrix of coefficients of deterministic term $\mathbf{D}_t$ $(s \times 1)$, which can contain a constant, a linear term, seasonal dummies, intervention dummies, or other regressors that we consider non-stochastic.

The process defined by the equation (1) can be written in *error correction* form

$$\Delta \mathbf{Y}_t = \mathbf{\Pi} \mathbf{Y}_{t-1} + \sum_{i=1}^{p-1} \mathbf{\Gamma}_i \Delta \mathbf{Y}_{t-i} + \mathbf{\Lambda} \mathbf{D}_t + \boldsymbol{\epsilon}_t\,, \quad t = 1, \ldots, T\,, \tag{2}$$

where $\mathbf{\Pi} = \sum_{i=1}^{p} \mathbf{\Phi}_i - \mathbf{I}$, $\mathbf{\Gamma}_i = -\sum_{j=i+1}^{p} \mathbf{\Phi}_j$. This error correction form of a VAR process is used in analysis of cointegration.

The basic idea of cointegration can be shown on 2 one-dimensional processes of order $I(1)$. We say that the processes X_t and Y_t are cointegrated if there exists any linear combination $aX_t + bY_t$ which is stationary.

Definition 4 Let $\mathbf{Y}_t$ be a n-dimensional process integrated of order 1. We call this process *cointegrated with a cointegrating vector* $\boldsymbol{\beta}$ ($\boldsymbol{\beta} \in \mathbb{R}^n, \boldsymbol{\beta} \neq \mathbf{0}$) if $\boldsymbol{\beta}'\mathbf{Y}_t$ can be made stationary by a suitable choice of its initial distribution.

If $n > 2$ then there may be two nonzero $n \times 1$ vectors $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ such that $\boldsymbol{\beta}_1'\mathbf{Y}_t$ and $\boldsymbol{\beta}_2'\mathbf{Y}_t$ are both stationary, where $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ are linearly independent. Indeed, there may be $r < n$ linearly independent cointegrating vectors. The *cointegrating rank* is the number of linearly independent cointegrating vectors and the space spanned by these vectors is the *cointegrating space*.

3 Estimation by Linear Regression

We use the method based on linear regression in this part of the article to obtain the estimate of the cointegrating vector. This method and tests of cointegration (Phillips-Quliaris-Hansen's tests) are described for example in Hamilton (1994) and Banerjee et al. (1993). We apply these tests on time series of some exchange rates.

NOK (Norwegian Crown), EUR (Euro), GBP (GB Pound), and USD (US Dollar) are the middle exchange rates of the CNB (the central bank of the Czech Republic) in 2002. The all null hypothesis of a unit root (Dickey-Fuller tests) were not rejected (see Table 1), so we can consider that these 4 rates form nonstationary time series of order $I(1)$.

Table 1: Tests of unit roots ($\alpha = 0.05$)

	$\hat{\rho}$	$T(\hat{\rho}-1)$	crit. value	$(\hat{\rho}-1)/\hat{\sigma}_{\hat{\rho}}$	crit. value
NOK	1.00037	0.093	−8	1.043	−1.95
EUR	0.99998	−0.006	−8	−0.073	−1.95
GBP	0.99982	−0.044	−8	−0.542	−1.95
USD	0.99939	−0.154	−8	−1.195	−1.95

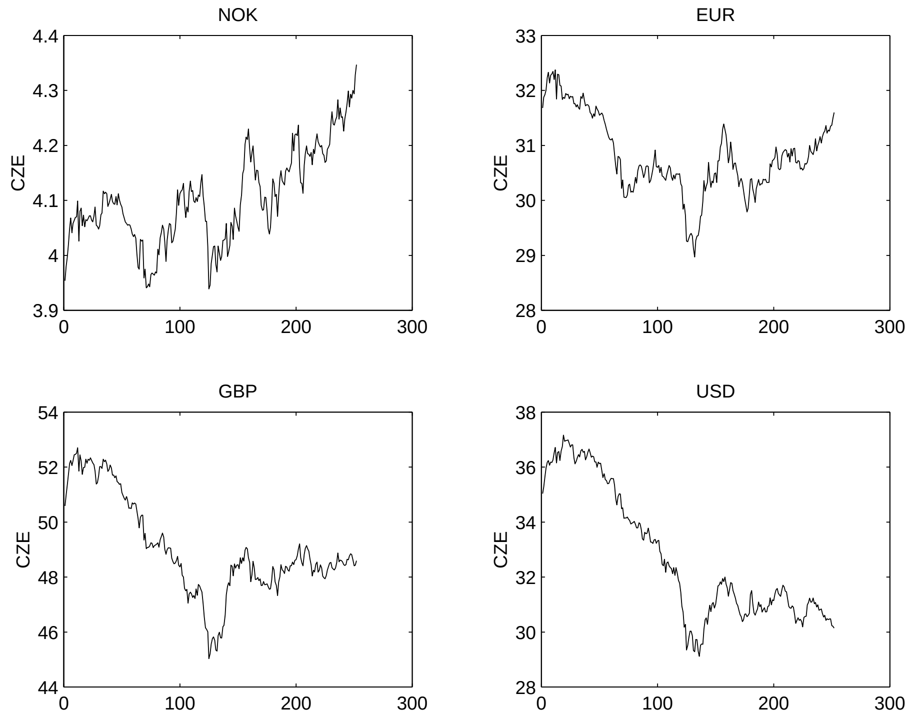

Figure 1: Exchange rates

Let Y_{1t} denote NOK, Y_{2t} EUR, Y_{3t} GBP, and Y_{4t} USD. The model considered is

$$Y_{1t} = \gamma_2 Y_{2t} + \gamma_3 Y_{3t} + \gamma_4 Y_{4t} + U_t \,,$$

giving parameter estimates $\hat{\gamma}_2 = 0.2018$ ($\hat{\sigma}_2 = 0.0015$), $\hat{\gamma}_3 = -0.0120$ ($\hat{\sigma}_3 = 0.0011$), $\hat{\gamma}_4 = -0.0467$ ($\hat{\sigma}_4 = 0.0003$), where $\hat{\sigma}_2, \hat{\sigma}_3, \hat{\sigma}_4$ are the standard errors of $\hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4$. We can write

$$Y_{1t} = 0.2018 Y_{2t} - 0.0120 Y_{3t} - 0.0467 Y_{4t} + U_t \,.$$

Consider the $AR(1)$ process applied on the residuals

$$U_t = \rho U_{t-1} + E_t \,.$$

This gives the estimates $\hat{\rho} = 0.9080$ ($\hat{\sigma}_{\hat{\rho}} = 0.0217$) and we write

$$U_t = 0.9080 U_{t-1} + E_t \,,$$

$$s^2 = \frac{1}{T-2} \sum_{t=2}^{T} \hat{E}_t^2 = 3.084 \cdot 10^{-4} \,,$$

$$\hat{c}_0 = 3.067 \cdot 10^{-4} \,, \quad \hat{c}_j = (T-1)^{-1} \sum_{t=j+2}^{T} \hat{E}_t \hat{E}_{t-1} \,,$$

$$\hat{\lambda}^2 = \hat{c}_0 + 2 \sum_{j=1}^{q} [1 - j/(q+1)] \hat{c}_j = 3.138 \cdot 10^{-4} \,, \quad (q = 1) \,.$$

Let Z_ρ be the test statistic (see Hamilton, 1994)

$$Z_\rho = (T-1)(\hat{\rho}-1) - (1/2)\cdot\{(T-1)^2\hat{\sigma}^2_{\hat{\rho}}/s^2\}\cdot\{\hat{\lambda}^2 - \hat{c}_0\} = -38.875\,.$$

If the significance level is $\alpha = 0.05$, the critical value of this test is -27.94 (see Hamilton, 1994). The null hypothesis of no cointegration is rejected by this test.

Let Z_t be the test statistic

$$Z_t = (\hat{c}_0/\hat{\lambda}^2)^{1/2}\cdot t - (1/2)\cdot\{(T-1)^2\hat{\sigma}^2_{\hat{\rho}}/s^2\}\cdot\{\hat{\lambda}^2 - \hat{c}_0\}/\hat{\lambda} = -4.247\,.$$

The critical value is -3.74 (see Hamilton, 1994) and the null hypothesis of no cointegration can be again rejected. The system of the middle rates is cointegrated (see Neubauer, 2004).

Let $\mathbf{Y}_t$ be a two-dimensional process. The estimate of the cointegrating vector we obtain by linear regression $Y_{1t} = a + \gamma Y_{2t} + U_t$. Obviously, we can also use the regression model $Y_{2t} = b + \delta Y_{1t} + V_t$. The estimate $\hat{\delta}$ is not simply the inverse of $\hat{\gamma}$, meaning that this two models give different estimates of the cointegrating vector. There can exist more linearly independent cointegrating vectors in a n-dimensional process. Using linear regression we can obtain only one. In the next part we focus on maximum likelihood estimates.

4 Maximum Likelihood Estimation

Granger's theorem (see for example Johansen, 1995) gives necessary and sufficient conditions for a VAR(p) process to be $I(1)$ and cointegrated. According to the rank of the matrix $\mathbf{\Pi}$ in the error correction form we define $H(r)$ as a model VAR(p) such that

$$\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'\,,$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $n\times r$ matrices. The reduced form error correction model is

$$\Delta\mathbf{Y}_t = \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{Y}_{t-1} + \sum_{i=1}^{p-1}\mathbf{\Gamma}_i\Delta\mathbf{Y}_{t-i} + \mathbf{\Lambda}\mathbf{D}_t + \boldsymbol{\epsilon}_t\,,\quad t = 1,\ldots,T\,, \tag{3}$$

where the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{\Gamma}_1,\ldots,\mathbf{\Gamma}_{p-1}$, $\mathbf{\Lambda}$, and $\mathbf{\Omega}$ vary freely.

Under the hypothesis

$$H(r) : \mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$$

the maximum likelihood estimator of $\boldsymbol{\beta}$ is found by the following procedure (see Johansen, 1995; Hamilton, 1994). First of all we solve the equation

$$|\lambda\mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}| = 0$$

for the eigenvalues $1 > \lambda_1 > \cdots \lambda_n > 0$ and eigenvectors $\mathbf{V} = (\mathbf{v}_1,\ldots,\mathbf{v}_n)$ which we normalize by $\mathbf{V}'\mathbf{S}_{11}\mathbf{V} = \mathbf{I}$. ($\mathbf{S}_{00}, \mathbf{S}_{01}, \mathbf{S}_{10}$, and $\mathbf{S}_{11}$ are $n\times n$ matrices defined in Johansen, 1995.) The cointegrating relations are estimated by

$$\hat{\boldsymbol{\beta}} = (\mathbf{v}_1,\ldots,\mathbf{v}_r)\,,$$

the maximized likelihood function is found from

$$L_{\max}^{-2/T} = |\mathbf{S}_{00}| \prod_{i=1}^{r} (1 - \lambda_i) \, . \tag{4}$$

The likelihood ratio test $Q(H(r)|H(n))$ for testing $H(r)$ in $H(n)$ we obtain by comparing two expressions (4) for r and n, then

$$Q(H(r)|H(n))^{-\frac{2}{T}} = \frac{|\mathbf{S}_{00}| \prod_{i=1}^{r}(1 - \lambda_i)}{|\mathbf{S}_{00}| \prod_{i=1}^{n}(1 - \lambda_i)} \, .$$

The logarithm of this expression is called the TRACE statistic

$$-2 \log Q(H(r)|H(n)) = -T \sum_{i=r+1}^{n} \log(1 - \lambda_i) \, . \tag{5}$$

The test statistic for testing $H(r)$ in $H(r+1)$ (MAX statistic) is given by

$$-2 \ln Q(H(r)|H(r+1)) = -T \ln(1 - \lambda_{r+1}). \tag{6}$$

The asymptotic distributions of (5) and (6) depend on the deterministic terms in the model (see Johansen, 1995).

We will find the maximum likelihood estimate of the cointegrating vector of the exchange rates NOK, EUR, GBP, and USD. We describe this 4-dimensional time series as a VAR(p) process. We use autocorrelation, cross-correlation functions and the Portmanteau statistic (see Johansen, 1995) to determine the parameter p in this model. The Portmanteau test statistic for VAR(1) is 268.96, the critical value (at the significance level $\alpha = 0.05$) is 277.14, so we consider the residuals of the VAR(1) to be uncorrelated and describe this time series as a VAR(1) process.

The estimate of the cointegrating vector is obtained by the procedure mentioned above. With respect to graphs of the exchange rates we do not include a constant into the model. All results are summarized in Table 2 and 3.

Table 2: The tests of cointegration - TRACE and MAX statistics

r	λ	TRACE	crit. value 0.05	crit. value 0.01
0	0.121	57.41	39.71	46.00
1	0.066	25.05	24.08	29.19
2	0.023	8.04	12.21	16.16
3	0.009	2.14	4.14	7.02
r	λ	MAX	crit. value 0.05	crit. value 0.01
0	0.121	32.36	23.97	28.94
1	0.066	17.01	17.64	21.91
2	0.023	5.90	11.14	14.92
3	0.009	2.14	4.14	7.02

Table 3: The estimates of the cointegrating vectors

	$\hat{\boldsymbol{\beta}}_1$	$\hat{\boldsymbol{\beta}}_2$	$\hat{\boldsymbol{\beta}}_3$	$\hat{\boldsymbol{\beta}}_4$
NOK	17.552	−8.635	1.012	−1.034
EUR	−2.679	3.682	1.463	0.448
GBP	−0.602	−1.817	−1.178	−0.635
USD	1.231	0.368	0.242	0.643

The TRACE and MAX statistics for $r = 0$ are greater than the critical values at significance level $\alpha = 0.05$ and 0.01. We suppose that there is one cointegrating relation ($r = 1$). The TRACE statistic is slightly greater than the critical value ($\alpha = 0.05$) but less than the critical value for $\alpha = 0.01$. The MAX statistic is less than its critical values for both significance levels. Following the given results we find one cointegrating vector (see table 3) $\hat{\boldsymbol{\beta}}' = (17.552, -2.679, -0.602, 1.231)$. After suitable normalization we get $\hat{\boldsymbol{\beta}}' = (1, -0.153, -0.034, 0.070)$. The cointegrating relation is determined by

$$\mathrm{NOK} = 0.153\,\mathrm{EUR} + 0.034\,\mathrm{GBP} - 0.070\,\mathrm{USD}\,. \tag{7}$$

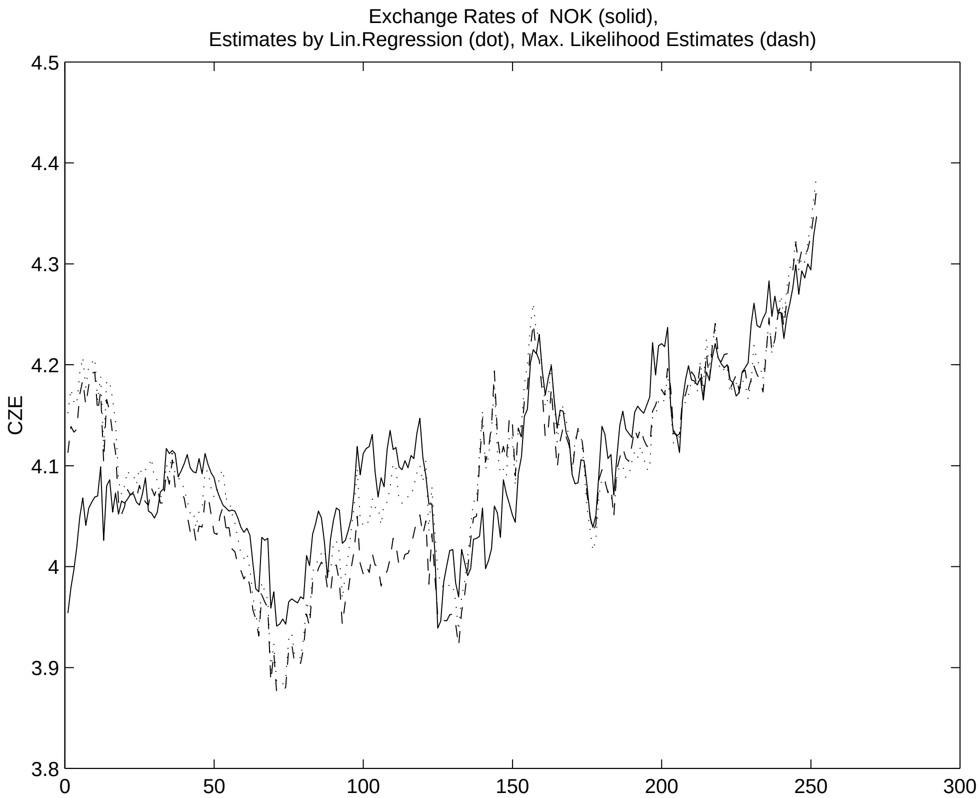

Figure 2: The estimates of exchange rates of NOK

We can see on the Figure 2 the exchange rates of NOK in 2002, the cointegrating relation estimated by the maximum likelihood (7) and the cointegrating relation obtained by linear regression

$$\mathrm{NOK} = 0.202\,\mathrm{EUR} - 0.012\,\mathrm{GBP} - 0.047\,\mathrm{USD}$$

to compare (see Section 3). The sum of square errors of NOK exchange rates and the cointegrating relation obtained by linear regression is 0.654, the sum of square errors of the maximum likelihood estimates is 0.816. From this point of view linear regression gives better estimation of this exchange rates.

All computations were made in environment MATLAB, a number of procedures were programmed for the purpose of numerical computing of the described methods.

5 Conclusion

In this paper were made analysis of 4-dimensional time series of the exchange rates of Norwegian crown, Euro, GB pound and USA dollar of the central bank of the Czech republic. All these time series were found non-stationary and $I(1)$ processes. The aim was to decide if this system was possible to consider cointegrated. The methods based on linear regression and maximum likelihood were used and the system of given exchange rates was found cointegrated by both methods. The estimates of the cointegrating vector were obtained and this methods were compared.

References

Banerjee, A., Dolado, J. J., Galbraith, J. W., and Hendry, D. F. (1993). *Co-integration, error-correction and the econometric analysis of non-stationary data* (1st ed.). Oxford: Oxford University Press.

Engle, R. F., and Granger, C. W. J. (1987). Co-integration and error correction: Representation, estimation, and testing. *Econometrica*, *55*, 251-276.

Hamilton, J. D. (1994). *Time series analysis* (1st ed.). Princeton: Princeton University Press.

Johansen, S. (1995). *Likelihood-based inference in cointegrated vector auto-regressive models* (1st ed.). Oxford: Oxford University Press.

Neubauer, J. (2004). Method of cointegration and exchange rates. *Summer School Datastat03, Proceedings, Folia Fac. Sci. Nat. Univ. Masaryk. Brunensis, Mathematica 15*, *15*, 241-255.

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