AUSTRIAN JOURNAL OF STATISTICS
Volume 34 (2005), Number 1, 25–38

Sequential Probability Ratio Test for Fuzzy Hypotheses Testing with Vague Data

Hamzeh Torabi and Javad Behboodian
Shiraz University, Iran

Abstract: In hypotheses testing, such as other statistical problems, we may confront imprecise concepts. One case is a situation in which both hypotheses and observations are imprecise.

In this paper, we redefine some concepts about fuzzy hypotheses testing, and then we give the sequential probability ratio test for fuzzy hypotheses testing with fuzzy observations. Finally, we give some applied examples.

Zusammenfassung: Bei Hypothesentests wie auch bei anderen statistischen Problemen könnten wir mit unpräzisen Konzepten konfrontiert sein. Ein Beispiel dafür ist die Situation in der beides, Hypothesen und Beobachtungen, unpräzise sind.

In diesem Artikel definieren wir einige Konzepte bei unscharfen Hypothesentests neu. Dann geben wir den sequentiellen Wahrscheinlichkeits-Quotiententest für unscharfes Hypothesentesten mit unscharfen Beobachtungen an. Zum Schluss führen wir einige angewandte Beispiele an.

Keywords: Critical Region, Type I and II Error Rates, Fuzzy Random Variable.

1 Introduction

Fuzzy set theory is a powerful and known tool for formulation and analysis of imprecise and subjective situations where exact analysis is either difficult or impossible. Some methods in descriptive statistics with vague data and some aspects of statistical inference is proposed in Kruse and Meyer (1987). Fuzzy random variables were introduced by Kwakernaak (1978), or Puri and Ralescu (1986) as a generalization of compact random sets, Kruse and Meyer (1987) and were developed by others such as Juninig and Wang (1989), Ralescu (1995), López-Díaz and Gil (1997), López-Díaz and Gil (1998), and Liu (2004).

In this paper, because of our main purpose (statistical inference about a parametric population with fuzzy data), we only consider and discuss fuzzy random variables associated with an ordinary random variable.

Decision making in classical statistical inference is based on crispness of data, random variables, exact hypotheses, decision rules and so on. As there are many different situations in which the above assumptions are rather unrealistic, there have been some attempts to analyze these situations with fuzzy set theory proposed by Zadeh (1965).

One of the primary purpose of statistical inference is to test hypotheses. In the traditional approach to hypotheses testing all the concepts are precise and well defined (see, e.g., Lehmann, 1994, Casella and Berger, 2002, and Shao, 1999). However, if we introduce vagueness into hypotheses, we face quite new and interesting problems. Arnold

(1996) considered statistical tests under fuzzy constraints on the type I and II errors. Testing fuzzy hypotheses was discussed by Arnold (1995) and Arnold (1998), Delgado et al. (1985), Saade and Schwarzlander (1990), Saade (1994), Watanabe and Imaizumi (1993), Taheri and Behboodian (1999), Taheri and Behboodian (2001), and Taheri and Behboodian (2002), and Grzegorzewski (2000) and Grzegorzewski (2002). Kruse and Meyer (1987), Taheri and Behboodian (2002) considered the problem of testing vague hypotheses in the presence of vague hypothesis. Up to now testing hypotheses with fuzzy data was considered by Casals et al. (1986), and Son et al. (1992). For more references about fuzzy testing problem see Taheri (2003). Also, for more details about ordinary sequential probability ratio test, see e.g. Hogg and Craig (1995) and Mood et al. (1974).

This paper is organized in the following way. In Section 2 we provide some definitions and preliminaries. Fuzzy hypotheses testing is defined in Section 3. The sequential probability ratio test for fuzzy hypotheses testing with vague data is introduced in Section 4. Finally, some applied examples are given in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A random variable (RV) X is a measurable function from $(\Omega, \mathcal{F}, \mathcal{P})$ to $(R, \mathcal{B}, \mathcal{P}_{\mathcal{X}})$, where P_X is the probability measure induced by X and is called the distribution of the RV X, i.e.,

$$P_X(A) = P(X \in A) = \int_{X \in A} dP\,, \quad A \in \mathcal{B}\,.$$

Using "the change of variable rule", (see e.g. Billingsley, 1995, p. 215 and 216, or Shao, 1999, p. 13), we have

$$P_X(A) = \int_A dPoX^{-1}(x) = \int_A dP_X(x)\,, \quad A \in \mathcal{B}\,.$$

If P_X is dominated by a σ-finite measure ν, i.e., $P_X \ll \nu$, then using the Radon-Nikodym theorem, (see e.g. Billingsley, 1995, p. 422 and 423, or Shao, 1999, p. 14), we have

$$P_X(A) = \int_A f(x)\, d\nu(x)\,,$$

where $f(x)$ is the Radon-Nikodym derivative of P_X with respect to ν and is called the probability density function (PDF) of X with respect to ν.

In statistical texts, the measure ν is usually a "counting measure" or a "Lebesgue measure"; hence $P_X(A)$ is calculated by $\sum_{x \in A} f(x)$ or $\int_A f(x)\, dx$, respectively.

Let $\mathcal{X} = \{x \in \mathbb{R} | f(x) > 0\}$. The set $\mathcal{X}$ is usually called "support" or "sample space" of X. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said a random sample of size n from a population with PDF $f(x)$, if the X_i's are independent distributed all with PDF $f(x)$ (X_i's are identically distributed). In this case, we have

$$f(\mathbf{x}) = f(x_1) \cdots f(x_n)\,, \quad \forall x_i \in \mathbb{R}\,,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is an observed value of $\mathbf{X}$.

In the following we present two definitions from the introduction of Casals et al. (1986), but in a slightly different way.

Definition 1 A fuzzy sample space $\tilde{\mathcal{X}}$ is a fuzzy partition (Ruspini partition) of $\mathcal{X}$, i.e., a set of fuzzy subsets of $\mathcal{X}$ whose membership functions are Borel measurable and satisfy the orthogonality constraint: $\sum_{\tilde{x}\in\tilde{\mathcal{X}}} \mu_{\tilde{x}}(x) = 1$, for each $x \in \mathcal{X}$.

Definition 2 A fuzzy random sample (FRS) of size n $\tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_n)$ associated with the PDF $f(x)$ is a measurable function from Ω to $\tilde{\mathcal{X}}^n$, whose PDF is given by

$$\tilde{f}(\tilde{x}_1, \ldots, \tilde{x}_n) = \tilde{P}(\tilde{\mathbf{X}} = \tilde{\mathbf{x}}) = \int_{\mathcal{X}^n} \prod_{i=1}^{n} \mu_{\tilde{x}_i}(x_i) f(x_i)\, d\nu(x_i)\,.$$

The density $\tilde{f}(\tilde{\mathbf{x}})$ is often called the fuzzy probability density function of $\tilde{X}$.

The above definition is according to Zadeh (1968). Note that using Fubini's theorem (see Billingsley, 1995, p. 233-234), we obtain independency of the $\tilde{X}_i$'s, i.e.,

$$\tilde{f}(\tilde{x}_1, \ldots, \tilde{x}_n) = \tilde{f}(\tilde{x}_1) \cdots \tilde{f}(\tilde{x}_n)\,, \quad \forall \tilde{x}_i \in \tilde{\mathcal{X}}\,,$$

where

$$\tilde{f}(\tilde{x}_i) = \int_{\mathcal{X}} \mu_{\tilde{x}_i}(x_i) f(x_i)\, d\nu(x_i)\,,$$

and $\tilde{f}(\tilde{x}_i)$ is the PDF of the fuzzy random variable (FRV) $\tilde{X}_i$, for each $i = 1, \ldots, n$. For each i, $\tilde{f}(\tilde{x}_i)$ really is a PDF on $\tilde{\mathcal{X}}$, because by the orthogonality of the $\mu_{\tilde{x}_i}$'s, we have

$$\begin{aligned}
\sum_{\tilde{x}_i\in\tilde{\mathcal{X}}} \tilde{f}(\tilde{x}_i) &= \sum_{\tilde{x}_i\in\tilde{\mathcal{X}}} \int_{\mathcal{X}} \mu_{\tilde{x}_i}(x_i) f(x_i)\, d\nu(x_i) \\
&= \int_{\mathcal{X}} f(x_i) \left(\sum_{\tilde{x}_i\in\tilde{\mathcal{X}}} \mu_{\tilde{x}_i}(x_i) \right) d\nu(x_i) \\
&= \int_{\mathcal{X}} f(x_i)\, d\nu(x_i) = 1\,.
\end{aligned}$$

Theorem 1 If g is a measurable function from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$, then $Y = g(\tilde{\mathbf{X}})$ is an ordinary random variable.

Proof: $\tilde{\mathbf{X}}$ is a measurable function from Ω to $\tilde{\mathcal{X}}^n$ and g is a measurable function from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$. Hence, $g(\tilde{\mathbf{X}}(\omega)) = go\tilde{\mathbf{X}}(\omega)$ is a composition of two measurable functions, therefore is measurable from Ω to $\mathbb{R}$ (see Billingsley, 1995, p. 182). ■

Note that using Theorem 1, we can define and use all related concepts for ordinary random variables, such as expectation, variance, etc.

Theorem 2 Let $\tilde{\mathbf{X}}$ be a fuzzy random sample with fuzzy sample space $\tilde{\mathcal{X}}^n$, and g be a measurable function from $\tilde{\mathcal{X}}^n$ to $\mathbb{R}$. The expectation of $g(\tilde{\mathbf{X}})$ is calculated by

$$\mathrm{E}\left[g(\tilde{\mathbf{X}})\right] = \sum_{\tilde{\mathbf{x}}\in\tilde{\mathcal{X}}^n} g(\tilde{\mathbf{x}}) \tilde{f}(\tilde{\mathbf{x}})\,.$$

Proof: Using the change of variable rule and the Radon-Nikodym theorem, we have

$$\begin{aligned} \mathrm{E}\left[g(\tilde{\mathbf{X}})\right] &= \int_{\Omega} g(\tilde{\mathbf{X}}(\omega))\, dP(\omega) \\ &= \int_{\tilde{\mathcal{X}}^n} g(\tilde{\mathbf{x}})\, dPo\tilde{\mathbf{X}}^{-1}(\tilde{\mathbf{x}}) \\ &= \sum_{\tilde{\mathbf{x}} \in \tilde{\mathcal{X}}^n} g(\tilde{\mathbf{x}}) \tilde{f}(\tilde{\mathbf{x}})\,. \ \blacksquare \end{aligned}$$

For more details about properties of ordinary RV's and their moments see, e.g., Ash and Doleans-Dade (2000), Billingsley (1995), Chung (2000), Feller (1968), or Ross (2002).

In this paper, we suppose that the PDF of the population is known but it has an unknown parameter $\theta \in \Theta$. In this case, we index $\tilde{f}$ by θ and write $\tilde{f}(\tilde{\mathbf{x}};\theta)$.

Example 1 Let X be a Bernoulli variable with parameter θ, i.e.,

$$f(x;\theta) = \theta^x (1-\theta)^{1-x}\,, \quad x = 0, 1\,, \quad 0 < \theta < 1\,.$$

We have $\mathcal{X} = \{0, 1\}$. Let $\tilde{x}_1$ and $\tilde{x}_2$ be two fuzzy subsets of $\mathcal{X}$ with membership functions

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 0.9\,, & x = 0 \\ 0.1\,, & x = 1\,, \end{cases} \qquad \text{and} \qquad \mu_{\tilde{x}_2}(x) = \begin{cases} 0.1\,, & x = 0 \\ 0.9\,, & x = 1\,. \end{cases}$$

Note that $\tilde{x}_1$ and $\tilde{x}_2$ are stated "approximately zero" and "approximately one" values, respectively. Here, the support of $\tilde{X}$ is $\tilde{\mathcal{X}} = \{\tilde{x}_1, \tilde{x}_2\}$. Therefore the PDF of $\tilde{X}$ is

$$\begin{aligned} \tilde{f}(\tilde{x};\theta) = \sum_{\mathcal{X}} \mu_{\tilde{x}_1}(x) f(x;\theta) &= \begin{cases} 0.9(1-\theta) + 0.1\theta\,, & \tilde{x} = \tilde{x}_1 \\ 0.1(1-\theta) + 0.9\theta\,, & \tilde{x} = \tilde{x}_2 \end{cases} \\ &= \begin{cases} 0.9 - 0.8\theta\,, & \tilde{x} = \tilde{x}_1 \\ 0.1 + 0.8\theta\,, & \tilde{x} = \tilde{x}_2\,. \end{cases} \end{aligned}$$

Let

$$Y = \begin{cases} 0.1\,, & \tilde{x} = \tilde{x}_1 \\ 0.9\,, & \tilde{x} = \tilde{x}_2\,. \end{cases}$$

Note that Y is a measurable function from $\tilde{\mathcal{X}}$ to $\mathbb{R}$ and therefore is a classical random variable. In the following, we calculate the mean and the variance of Y. The PDF of Y is

$$f_Y(y;\theta) = \begin{cases} 0.9 - 0.8\theta\,, & y = 0.1 \\ 0.1 + 0.8\theta\,, & y = 0.9\,. \end{cases}$$

Therefore, the expectation and the variance of Y are

$$\begin{aligned} \mathrm{E}(Y) &= 0.1(0.9 - 0.8\theta) + 0.9(0.1 + 0.8\theta) = 0.18 + 0.64\theta \\ \mathrm{E}(Y^2) &= 0.01(0.9 - 0.8\theta) + 0.81(0.1 + 0.8\theta) = 0.09 + 0.64\theta \\ \mathrm{Var}(Y) &= 0.09 + 0.64\theta - (0.18 + 0.64\theta)^2 = 0.0576 + 0.4096\theta - 0.4096\theta^2\,. \end{aligned}$$

3 Fuzzy Hypotheses Testing

In this section we introduce concepts about fuzzy hypotheses testing (FHT).

Definition 3 Any hypothesis of the form "$H : \theta$ is $H(\theta)$" is called a fuzzy hypothesis, where "$H : \theta$ is $H(\theta)$" implies that θ is in a fuzzy set of Θ, the parameter space, with membership function $H(\theta)$, i.e., a function from Θ to $[0, 1]$.

Note that the ordinary hypothesis $H : \theta \in \Theta$ is a fuzzy hypothesis with membership function $H(\theta) = 1$ at $\theta \in \Theta$, and zero otherwise, i.e., the indicator function of the crisp set Θ.

Example 2 Let θ be the parameter of a Bernoulli distribution. Consider the following function:

$$H(\theta) = \begin{cases} 2\theta\,, & 0 < \theta < 1/2 \\ 2 - 2\theta\,, & 1/2 \le \theta < 1\,. \end{cases}$$

The hypothesis "$H : \theta$ is $H(\theta)$" is a fuzzy hypothesis and it means that "θ is approximately 1/2".

In FHT with fuzzy data, the main problem is testing

$$\begin{cases} H_0 : \theta \text{ is } H_0(\theta) \\ H_1 : \theta \text{ is } H_1(\theta) \end{cases} \qquad (1)$$

according to a fuzzy random sample $\tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_n)$ from a parametric fuzzy population with PDF $\tilde{f}(\tilde{x}; \theta)$. In the following we give some definitions in FHT theory with fuzzy data.

Definition 4 The normalized membership function of $H_j(\theta)$ is defined by

$$H_j^*(\theta) = \frac{H_j(\theta)}{\int_\Theta H_j(\theta)\, d\theta}\,, \quad j = 0, 1$$

providing to $\int_\Theta H_j(\theta)\, d\theta < \infty$. Replace integration by summation in discrete cases.

Note that the normalized membership function is not necessarily a membership function, i.e., it may be greater than 1 for some values of θ.

In FTH with fuzzy data, like in traditional hypotheses testing, we must give a test function $\tilde{\Phi}(\tilde{\mathbf{X}})$, which is defined in the following.

Definition 5 Let $\tilde{\mathbf{X}}$ be a FRS with PDF $\tilde{f}(\tilde{\mathbf{x}}; \theta)$. $\tilde{\Phi}(\tilde{\mathbf{X}})$ is called a fuzzy test function, if it is the probability of rejecting H_0 provided $\tilde{\mathbf{X}} = \tilde{\mathbf{x}}$ is observed.

Definition 6 Let the FRV $\tilde{X}$ have PDF $\tilde{f}(\tilde{x}; \theta)$. Under $H_j(\theta)$, $j = 0, 1$, the weighted probability density function (WPDF) of $\tilde{X}$ is defined by

$$\tilde{f}_j(\tilde{x}) = \int_\Theta H_j^*(\theta) \tilde{f}(\tilde{x}; \theta)\, d\theta\,,$$

i.e., the expected value of $\tilde{f}(\tilde{x}; \theta)$ over $H_j^*(\theta)$, $j = 0, 1$. If $\tilde{\mathbf{X}}$ is a fuzzy random sample from $\tilde{f}(\cdot; \theta)$, then the joint WPDF of $\tilde{\mathbf{X}}$ is

$$\tilde{f}_j(\tilde{\mathbf{x}}) = \prod_{i=1}^{n} \tilde{f}_j(\tilde{x}_i)\,.$$

Remark 1 $\tilde{f}_j(\tilde{x})$ is a PDF, since $\tilde{f}_j(\tilde{x})$ is nonnegative and

$$\begin{aligned}\sum_{\tilde{x}} \tilde{f}_j(\tilde{x}) &= \sum_{\tilde{x}} \int_\Theta H_j^*(\theta)\tilde{f}(\tilde{x};\theta)\, d\theta \\ &= \int_\Theta H_j^*(\theta) \left(\sum_{\tilde{x}} \tilde{f}(\tilde{x};\theta)\right) d\theta \\ &= \int_\Theta H_j^*(\theta)\, d\theta = 1\,.\end{aligned}$$

Hence, $\tilde{f}_j(\tilde{x}_1, \ldots, \tilde{x}_n)$ is also a joint PDF.

Remark 2 If H_j is the crisp hypothesis $H_j : \theta = \theta_j$, then $\tilde{f}_j(\tilde{x}) = \tilde{f}(\tilde{x};\theta_j)$ and $\tilde{f}_j(\tilde{x}_1, \ldots, \tilde{x}_n) = \tilde{f}(\tilde{x}_1, \ldots, \tilde{x}_n;\theta_j)$, $j = 0, 1$.

Definition 7 Let $\tilde{\Phi}(\tilde{\mathbf{X}})$ be a fuzzy test function. The probability of type I and II errors of $\tilde{\Phi}(\tilde{\mathbf{X}})$ for the fuzzy testing problem (1) is defined by $\alpha_{\tilde{\Phi}} = \mathrm{E}_0[\tilde{\Phi}(\tilde{\mathbf{X}})]$ and $\beta_{\tilde{\Phi}} = 1 - \mathrm{E}_1[\tilde{\Phi}(\tilde{\mathbf{X}})]$, respectively, where $\mathrm{E}_j[\tilde{\Phi}(\tilde{\mathbf{X}})]$ is the expected value of $\tilde{\Phi}(\tilde{\mathbf{X}})$ over the joint WPDF $\tilde{f}_j(\tilde{\mathbf{x}})$, $j = 0, 1$.

Note that in the case of testing a simple crisp hypothesis against simple crisp alternative, i.e.,

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta = \theta_1 \end{cases}$$

and for crisp observations, the above definition of $\alpha_{\tilde{\Phi}}$ and $\beta_{\tilde{\Phi}}$ gives the classical probability of errors.

Regarding to definitions of error sizes, it is concluded that fuzzy hypotheses testing (1) is equivalent to the following ordinary hypotheses testing

$$\begin{cases} H_0' : \tilde{\mathbf{X}} \sim \tilde{f}_0 \\ H_1' : \tilde{\mathbf{X}} \sim \tilde{f}_1 \end{cases} \tag{2}$$

Definition 8 A fuzzy testing problem with a test function $\tilde{\Phi}$ is said to be a test of (significance) level α, if $\alpha_{\tilde{\Phi}} \leq \alpha$, where $\alpha \in [0, 1]$. We call $\alpha_{\tilde{\Phi}}$ the size of $\tilde{\Phi}$.

4 Sequential Probability Ratio Test for FHT

In this section, first, we define the sequential probability ratio test(SPRT) for the ordinary simple hypotheses testing with crisp observations and then concerning Section 3, we extend the SPRT to the FHT with fuzzy observations.

Consider testing a simple null hypothesis against a simple alternative hypothesis. In other words, suppose a sample can be drawn from one of two known distributions and it is desired to test that the sample came from one distribution against the possibility that it came from the other. If $X_1, X_2, \cdots$ denote the iid RV's, we want to test $H_0 : X_i \sim f_0(\cdot)$ versus $H_1 : X_i \sim f_1(\cdot)$. For a sample of size m, the Neyman-Pearson criterion rejects H_0 if $R_m(\mathbf{x}) = L_0(\mathbf{x})/L_1(\mathbf{x}) < k$, for some constant $k > 0$, where $\mathbf{x} = (x_1, \ldots, x_m)$,

$L_j(\mathbf{x}) = \prod_{i=1}^{m} f_j(x_i)$, $j = 0, 1$. Compute sequentially $R_1, R_2, \ldots$. For fixed k_0 and k_1 satisfying $0 < k_0 < k_1$, adopt the following procedure: Take observation x_1 and compute R_1; if $R_1 \leq k_0$, reject H_0; if $R_1 \geq k_1$, accept H_0; and if $k_0 < R_1 < k_1$, take observation x_2, and compute R_2; if $R_2 \leq k_0$, reject H_0; if $R_2 \geq k_1$, accept H_0; and if $k_0 < R_2 < k_1$, take observation x_3, etc. The idea is to continue sampling as long as $k_0 < R_j < k_1$ and stop as soon as $R_m \leq k_0$ or $R_m \geq k_1$, rejecting H_0 if $R_m \leq k_0$ and accepting H_0 if $R_m \geq k_1$. The critical region of the described sequential test can be defined as $C = \bigcup_{n=1}^{\infty} C_n$, where

$$C_n = \{(x_1, \ldots, x_n) | k_0 < R_j(x_1, \ldots, x_j) < k_1 \,,\; j = 1, \ldots, n-1 \,,\; R_n(x_1, \ldots, x_n) \leq k_0\} \,.$$

Similarly, the acceptance region can be defined as $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{(x_1, \ldots, x_n) | k_0 < R_j(x_1, \ldots, x_j) < k_1 \,,\; j = 1, \ldots, n-1 \,,\; R_n(x_1, \ldots, x_n) \geq k_1\} \,.$$

Definition 9 For fixed k_0 and k_1, a test as described above is defined to be a sequential probability ratio test (SPRT). Therefore for the SPRT, the probability of type I and II errors is calculated by $\alpha = \sum_{n=1}^{\infty} \int_{C_n} L_0(\mathbf{x})\, d\mathbf{x}$, and $\beta = \sum_{n=1}^{\infty} \int_{A_n} L_1(\mathbf{x})\, d\mathbf{x}$, respectively.

In the following, we briefly state some results about the classical SPRT without proofs. For more details see Mood et al. (1974) or Hogg and Craig (1995).

Let k_0 and k_1 be defined so that the SPRT has fixed probabilities of type I and II errors α and β. Then k_0 and k_1 can be approximated by $k_0' = \alpha/(1-\beta)$ and $k_1' = (1-\alpha)/\beta$, respectively. If α' and β' are the error sizes of the SPRT defined by k_0 and k_1, then $\alpha' + \beta' \leq \alpha + \beta$.

If $z_i = \log(f_0(x_i)/f_1(x_i))$, an equivalent test to the SPRT is given by the following: continue sampling as long as $\log(k_0) < \sum_{i=1}^{m} z_i < \log(k_1)$, and stop sampling when $\sum_{i=1}^{m} z_i \leq \log(k_0)$ (and reject H_0) or $\sum_{i=1}^{m} z_i \geq \log(k_1)$ (and accept H_0).

Let N be the RV denoting the sample size of the SPRT. The SPRT with error sizes α and β minimizes both $\mathrm{E}[N|H_0 \text{ true }]$ and $\mathrm{E}[N|H_1 \text{ true }]$ among all tests (sequential or not) which satisfy

$$P(H_0 \text{ rejected } | H_0 \text{ true}) \leq \alpha \,, \qquad \text{and} \quad P(H_0 \text{ accepted } | H_0 \text{ false}) \leq \beta \,.$$

Using Wald's equation we obtain $\mathrm{E}[N] = \mathrm{E}[Z_1 + \cdots + Z_N]/\mathrm{E}[Z_1]$. But $\mathrm{E}[Z_1 + \cdots + Z_N] \approx \rho \log(k_0) + (1-\rho)\log(k_1)$, where $\rho = P(\text{reject } H_0)$. Hence,

$$\mathrm{E}[N|H_0 \text{ true }] \approx \frac{\alpha \log[\alpha/(1-\beta)] + (1-\alpha)\log[(1-\alpha)/\beta]}{\mathrm{E}[Z_1|H_0 \text{ true}]} \,,$$

$$\mathrm{E}[N|H_1 \text{ true}] \approx \frac{(1-\beta)\log[\alpha/(1-\beta)] + \beta \log[(1-\alpha)/\beta]}{\mathrm{E}[Z_1|H_1 \text{ true }]} \,.$$

Now, we are ready to state the SPRT for fuzzy hypotheses testing with vague data.

Definition 10 Let $\tilde{X}_1, \tilde{X}_2, \ldots$ be an iid sequence of FRV's from a population with PDF $\tilde{f}(\cdot;\theta)$. We propose to consider testing

$$\begin{cases} H_0' : \tilde{\mathbf{X}} \sim \tilde{f}_0 \\ H_1' : \tilde{\mathbf{X}} \sim \tilde{f}_1 \end{cases}$$

as a SPRT for fuzzy hypotheses testing (1), in which $\tilde{f}_j(\tilde{x})$ is the WPDF of $\tilde{f}(\tilde{x};\theta)$ under $H_j^*(\theta)$, $j = 0, 1$ (see Definition 6). Thus, the critical region of the described SPRT for fuzzy hypotheses testing (1) is defined as $C = \bigcup_{n=1}^{\infty} C_n$, where

$$C_n = \{(\tilde{x}_1, \dots, \tilde{x}_n) | k_0 < \tilde{R}_j(\tilde{x}_1, \dots, \tilde{x}_j) < k_1 \, , \, j = 1, \dots, n-1 \, , \, \tilde{R}_n(\tilde{x}_1, \dots, \tilde{x}_n) \le k_0\} \, .$$

Similarly, the acceptance region can be defined as $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{(\tilde{x}_1, \dots, \tilde{x}_n) | k_0 < \tilde{R}_j(\tilde{x}_1, \dots, \tilde{x}_j) < k_1 \, , \, j = 1, \dots, n-1 \, , \, \tilde{R}_n(\tilde{x}_1, \dots, \tilde{x}_n) \ge k_1\} \, ,$$

in which

$$\begin{aligned} \tilde{R}_m(\tilde{x}_1, \dots, \tilde{x}_m) &= \tilde{f}_0(\tilde{\mathbf{x}}) / \tilde{f}_1(\tilde{\mathbf{x}}) \\ &= \prod_{i=1}^{m} \left[\tilde{f}_0(\tilde{x}_i) / \tilde{f}_1(\tilde{x}_i) \right] \\ &= \prod_{i=1}^{m} \left[\int_\Theta H_0^*(\theta) \tilde{f}(\tilde{x}_i; \theta) \, d\theta \Big/ \int_\Theta H_1^*(\theta) \tilde{f}(\tilde{x}_i; \theta) \, d\theta \right] \, . \end{aligned}$$

Regarding to the definition of WPDF, α, β and other related concepts, all results of the ordinary SPRT are satisfied for this case, of course with the following modifications. In this case, we have

$$\begin{aligned} k_0' &= \frac{\alpha_{\tilde{\Phi}}}{1 - \beta_{\tilde{\Phi}}} \, , \qquad k_1' = \frac{1 - \alpha_{\tilde{\Phi}}}{\beta_{\tilde{\Phi}}} \\ Z_i &= \log \frac{\int_\Theta H_0^*(\theta) \tilde{f}(\tilde{X}_i; \theta) \, d\theta}{\int_\Theta H_1^*(\theta) \tilde{f}(\tilde{X}_i; \theta) \, d\theta} \, , \quad i = 1, 2, \dots \end{aligned}$$

and

$$\mathrm{E}[Z_i | H_j \text{ true}] = \sum_{\tilde{x}_i \in \tilde{\mathcal{X}}} \log \frac{\int_\Theta H_0^*(\theta) \tilde{f}(\tilde{x}_i; \theta) \, d\theta}{\int_\Theta H_1^*(\theta) \tilde{f}(\tilde{x}_i; \theta) \, d\theta} \tilde{f}_j(\tilde{x}_i) \, , \quad j = 0, 1 \, .$$

Hence,

$$\begin{aligned} \mathrm{E}[N | H_0 \text{ true}] &\approx \frac{\alpha_{\tilde{\Phi}} \log(k_0') + (1 - \alpha_{\tilde{\Phi}}) \log(k_1')}{\mathrm{E}[Z_1 | H_0 \text{ true}]} \\ \mathrm{E}[N | H_1 \text{ true}] &\approx \frac{(1 - \beta_{\tilde{\Phi}}) \log(k_0') + \beta_{\tilde{\Phi}} \log(k_1')}{\mathrm{E}[Z_1 | H_1 \text{ true}]} \, . \end{aligned}$$

Note that Z_i is an ordinary RV.

5 Some Examples

In this section, we present four important examples to clarify the theoretical discussions so far.

Example 3 Let $X_1, X_2, \ldots$ be a sequence of iid RV's from Bernoulli(θ), $0 < \theta < 1$. We want to test

$$\begin{cases} H_0 : \theta \approx \theta_0 \\ H_1 : \theta \approx \theta_1 \end{cases}$$

where

$$H_j(\theta) = \theta^{\alpha_j}(1-\theta)\,, \quad \forall \theta \in (0,1)\,,\ j = 0, 1,\ \text{for } \alpha_0 = 7\,,\ \alpha_1 = 1/7$$

according to two fuzzy data (fuzzy subsets of $\mathcal{X} = \{0, 1\}$) $\tilde{x}_I$ and $\tilde{x}_{II}$ where their membership functions are defined by

$$\mu_{\tilde{x}_I}(x) = \begin{cases} 0.9\,,\ x = 0 \\ 0.1\,,\ x = 1\,, \end{cases} \qquad \mu_{\tilde{x}_{II}}(x) = \begin{cases} 0.1\,,\ x = 0 \\ 0.9\,,\ x = 1\,. \end{cases}$$

The normalized membership function of $H_j(\theta)$ is

$$H_j^*(\theta) = (\alpha_j + 2)(\alpha_j + 1)\theta^{\alpha_j}(1-\theta)\,.$$

If we denote this FRV, its fuzzy observation and its PDF by $\tilde{X}$, $\tilde{x}$, and $\tilde{f}(\tilde{x};\theta)$, respectively, then using Example 1, we have

$$\tilde{f}(\tilde{x};\theta) = \begin{cases} 0.9 - 0.8\theta\,,\ \tilde{x} = \tilde{x}_I \\ 0.1 + 0.8\theta\,,\ \tilde{x} = \tilde{x}_{II}\,. \end{cases}$$

It is easy to show that

$$\tilde{f}_0(\tilde{x}) = \begin{cases} 0.26\,,\ \tilde{x} = \tilde{x}_I \\ 0.74\,,\ \tilde{x} = \tilde{x}_{II}\,, \end{cases} \qquad \tilde{f}_1(\tilde{x}) = \begin{cases} 0.609\,,\ \tilde{x} = \tilde{x}_I \\ 0.391\,,\ \tilde{x} = \tilde{x}_{II}\,. \end{cases}$$

Hence, for $i = 1, 2, \ldots$, we obtain

$$z_i = \log \frac{\tilde{f}_0(\tilde{x}_i)}{\tilde{f}_1(\tilde{x}_i)} = \begin{cases} -0.85114\,,\ \tilde{x}_i = \tilde{x}_I \\ 0.63794\,,\ \tilde{x}_i = \tilde{x}_{II}\,. \end{cases}$$

Assume that $\alpha_{\tilde{\Phi}} = 0.1$ and $\beta_{\tilde{\Phi}} = 0.01$. Then we obtain $\log(k_0') = -2.2925$, $\log(k_1') = 4.4998$, $\mathrm{E}[Z_i|H_0 \text{ true }] = 0.25078$, and $\mathrm{E}[Z_i|H_1 \text{ true}] = -0.26891$. Hence, $\mathrm{E}[N|H_0 \text{ true}] = 15.235$, then we take $n = 16$ and $\mathrm{E}[N|H_1 \text{ true}] = 8.273$, thus we take $n = 9$.

Example 4 Let $X_1, X_2, \ldots$ be a sequence of iid RV's from a N(μ, σ^2) population, i.e.

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,, \quad x, \mu \in \mathbb{R}\,,\ \sigma > 0\,.$$

We want to test

$$\begin{cases} H_0 : \mu \approx \mu_0 \\ H_1 : \mu \approx \mu_1 \end{cases}$$

with membership functions

$$H_j(\mu) = \exp\left(-\frac{(\mu-\mu_j)^2}{2\sigma_0^2}\right)\,, \quad j = 0, 1\,, \quad \mu \in \mathbb{R}\,,\ \sigma_0 > 0\,,$$

using the SPRT, according to three fuzzy data (fuzzy subsets of $\mathcal{X} = (-\infty, +\infty)$) $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$, where their membership functions are defined by

$$\mu_{\tilde{x}_I}(x) = \begin{cases} 1 - e^{-x^2/2}\,, & x < 0 \\ 0\,, & x \geq 0\,, \end{cases}$$
$$\mu_{\tilde{x}_{II}}(x) = e^{-x^2/2}\,,\ x \in \mathbb{R}\,,$$
$$\mu_{\tilde{x}_{III}}(x) = \begin{cases} 0\,, & x < 0 \\ 1 - e^{-x^2/2}\,, & x \geq 0\,. \end{cases}$$

The fuzzy subsets $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$ can be interpreted as the values of “very small”, “near to zero”, and “very large”.

Note that the μ’s are measurable and satisfy the orthogonality constraint (see Definitions 1 and 2).

The normalized membership function of $H_j(\theta)$ is

$$H_j^*(\mu) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{(\mu-\mu_j)^2}{2\sigma_0^2}\right)\,, \quad j = 0, 1\,,\ \mu \in \mathbb{R}\,,\ \sigma_0 > 0\,.$$

Denote this FRV, its fuzzy observation and its PDF by $\tilde{X}$, $\tilde{x}$, and $\tilde{f}(\tilde{x};\theta)$, respectively. Let $\mu_0 = 0$, $\mu_1 = 1$, $\sigma^2 = 4$, and $\sigma_0^2 = 0.5$. It is easy to show that

$$\tilde{f}_0(\tilde{x}) = \begin{cases} 0.3796\,, & \tilde{x} = \tilde{x}_I \\ 0.2408\,, & \tilde{x} = \tilde{x}_{II} \\ 0.3796\,, & \tilde{x} = \tilde{x}_{III}\,, \end{cases} \qquad \tilde{f}_1(\tilde{x}) = \begin{cases} 0.2907\,, & \tilde{x} = \tilde{x}_I \\ 0.2339\,, & \tilde{x} = \tilde{x}_{II} \\ 0.4754\,, & \tilde{x} = \tilde{x}_{III}\,. \end{cases}$$

Hence, for $i = 1, 2, \ldots$, we get

$$z_i = \log\frac{\tilde{f}_0(\tilde{x}_i)}{\tilde{f}_1(\tilde{x}_i)} = \begin{cases} 0.2668\,, & \tilde{x}_i = \tilde{x}_I \\ 0.0291\,, & \tilde{x}_i = \tilde{x}_{II} \\ -0.2250\,, & \tilde{x}_i = \tilde{x}_{III}\,. \end{cases}$$

Suppose that $\alpha_{\tilde{\Phi}} = \beta_{\tilde{\Phi}} = 0.1$. We have $\log(k_0') = -2.19722$, $\log(k_1') = 2.19772$, $\mathrm{E}[Z_i|H_0 \text{ true}] = 0.02287$, and $\mathrm{E}[Z_i|H_1 \text{ true}] = -0.02261$. Thus, $\mathrm{E}[N|H_0 \text{ true}\,] = 76.815$, then in this case, we take $n = 77$ and $\mathrm{E}[N|H_1 \text{ true}] = 77.658$, thus we must take $n = 78$.

Example 5 Let $X_1, X_2, \ldots$ be a sequence of iid RV’s from an Exponential population with mean θ, i.e.

$$f(x;\theta) = \frac{1}{\theta}\exp\left(-x/\theta\right)\,, \quad x, \theta > 0\,.$$

We want to test

$$\begin{cases} H_0 : \theta \approx 1/2 \\ H_1 : \theta \approx 3/2\,, \end{cases}$$

where the membership functions $H_0(\theta)$ and $H_1(\theta)$ are defined by

$$H_0(\theta) = \begin{cases} 2\theta\,, & 0 < \theta \leq 1/2 \\ 2 - 2\theta\,, & 1/2 < \theta < 1 \\ 0\,, & \text{otherwise,} \end{cases} \qquad H_1(\theta) = \begin{cases} 2\theta - 2\,, & 1 < \theta \leq 3/2 \\ 4 - 2\theta\,, & 3/2 < \theta < 2 \\ 0\,, & \text{otherwise} \end{cases}$$

using the SPRT, according to three fuzzy data, fuzzy subsets of $\mathcal{X} = (0, +\infty)$, $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$, where their membership functions are defined by

$$\mu_{\tilde{x}_I}(x) = \begin{cases} e^{-x}\,, & 0 < x < 1 \\ 0\,, & x \geq 1\,, \end{cases} \qquad \mu_{\tilde{x}_{II}}(x) = \begin{cases} 1 - e^{-x}\,, & 0 < x < 1 \\ 1\,, & 1 \leq x < 2 \\ e^{-x}\,, & x \geq 2\,, \end{cases}$$

$$\mu_{\tilde{x}_{III}}(x) = \begin{cases} 0\,, & 0 < x < 2 \\ 1 - e^{-x}\,, & x \geq 2\,. \end{cases}$$

We can interpret the fuzzy subsets $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$ as the values of "near to zero", "near to 3/2", and "very large". Note that μ's are measurable and satisfy the orthogonality constraint.

The normalized membership function of $H_j(\theta)$ is

$$H_j^*(\theta) = 2H_j(\theta)\,, \quad j = 0, 1\,.$$

Denote this FRV, its fuzzy observation and its PDF by $\tilde{X}$, $\tilde{x}$, and $\tilde{f}(\tilde{x};\theta)$, respectively.

It can be shown that

$$\tilde{f}(\tilde{x};\theta) = \begin{cases} \frac{1}{\theta+1}\left[1 - e^{-(\theta+1)/\theta}\right], & \tilde{x} = \tilde{x}_1 \\ 1 - \frac{1}{\theta+1}\left[1 - e^{-(\theta+1)/\theta}\right] - e^{-2/\theta} + \frac{1}{\theta+1}e^{-2(\theta+1)/\theta}\,, & \tilde{x} = \tilde{x}_2 \\ e^{-2/\theta} - \frac{1}{\theta+1}e^{-2(\theta+1)/\theta}\,, & \tilde{x} = \tilde{x}_3 \end{cases}$$

and hence,

$$\tilde{f}_0(\tilde{x}) = \begin{cases} 0.648\,, & \tilde{x} = \tilde{x}_I \\ 0.326\,, & \tilde{x} = \tilde{x}_{II} \\ 0.026\,, & \tilde{x} = \tilde{x}_{III}\,, \end{cases} \qquad \tilde{f}_1(\tilde{x}) = \begin{cases} 0.328\,, & \tilde{x} = \tilde{x}_I \\ 0.425\,, & \tilde{x} = \tilde{x}_{II} \\ 0.247\,, & \tilde{x} = \tilde{x}_{III}\,. \end{cases}$$

Thus, for $i = 1, 2, \ldots$, we obtain

$$z_i = \log \frac{\tilde{f}_0(\tilde{x}_i)}{\tilde{f}_1(\tilde{x}_i)} = \begin{cases} 0.681\,, & \tilde{x}_i = \tilde{x}_I \\ -0.265\,, & \tilde{x}_i = \tilde{x}_{II} \\ -2.251\,, & \tilde{x}_i = \tilde{x}_{III}\,. \end{cases}$$

If $\alpha_{\tilde{\Phi}} = \beta_{\tilde{\Phi}} = 0.05$ we get $\log(k_0') = -2.9444$, $\log(k_1') = 2.9444$, $\mathrm{E}[Z_i|H_0 \text{ true}] = 0.296$, and $\mathrm{E}[Z_i|H_1 \text{ true}] = -0.445$. Hence, $\mathrm{E}[N|H_0 \text{ true}] = 8.95$, and we therefore take $n = 9$, whereas $\mathrm{E}[N|H_1 \text{ true}] = 5.95$ and we take $n = 6$.

Example 6 Let X be a RV with PDF

$$f(x;\theta) = 2\theta x + 2(1-\theta)(1-x)\,, \quad 0 < x < 1\,,\ 0 < \theta < 1\,.$$

We want to test

$$\begin{cases} H_0 : \theta \approx 1/4 & (\theta \text{ is approximately } 1/4) \\ H_0 : \theta \approx 3/4 & (\theta \text{ is approximately } 3/4)\,. \end{cases}$$

where the membership functions are defined as

$$H_0(\theta) = \begin{cases} 4\theta\,, & 0 < \theta \leq 1/4 \\ 2 - 4\theta\,, & 1/4 < \theta < 1/2 \\ 0\,, & \text{otherwise,} \end{cases} \qquad H_1(\theta) = \begin{cases} 4\theta - 2\,, & 1/2 < \theta \leq 3/4 \\ 4 - 4\theta\,, & 3/4 < \theta < 1 \\ 0\,, & \text{otherwise,} \end{cases}$$

according to three fuzzy data (fuzzy subsets of $\mathcal{X} = (0,1)$) $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$, with membership functions

$$\mu_{\tilde{x}_I}(x) = \begin{cases} 0.8 - 0.8x\,, & 0 < x \le 1/2 \\ 0\,, & 1/2 < x < 1\,. \end{cases} \qquad \mu_{\tilde{x}_{II}}(x) = \begin{cases} 0.2 + 0.8x\,, & 0 < x \le 1/2 \\ 1 - 0.8x\,, & 1/2 < x < 1\,. \end{cases}$$

$$\mu_{\tilde{x}_{III}}(x) = \begin{cases} 0\,, & 0 < x \le 1/2 \\ 0.8x\,, & 1/2 < x < 1\,. \end{cases}$$

We can interpret the fuzzy subsets $\tilde{x}_I$, $\tilde{x}_{II}$, and $\tilde{x}_{III}$ as the values of "near to zero", "near to 0.5", and "near to 1". It is clear that all μ's are measurable and satisfy the orthogonality constraint of Definition 1.

If we denote this FRV, its fuzzy observation and its PDF by $\tilde{X}$, $\tilde{x}$, and $\tilde{f}(\tilde{x};\theta)$, respectively, then using Definition 2, we have

$$\tilde{f}(\tilde{x};\theta) = \int_0^1 \mu_{\tilde{x}}(x) f(x;\theta)\, dx = \begin{cases} 0.467 - 0.333\theta\,, & \tilde{x} = \tilde{x}_I \\ 0.400\,, & \tilde{x} = \tilde{x}_{II} \\ 0.133 + 0.333\theta\,, & \tilde{x} = \tilde{x}_{III}\,. \end{cases}$$

On the other hand, the normalized membership function of $H_j(\theta)$ is

$$H_j^*(\theta) = 4H_j(\theta)\,, \quad j = 0,1\,.$$

It is easy to show that the WPDF's of $\tilde{X}$ are

$$\tilde{f}_0(\tilde{x}) = \int_0^1 H_0^*(\theta)\tilde{f}(\tilde{x};\theta)\, d\theta = \begin{cases} 0.383\,, & \tilde{x} = \tilde{x}_I \\ 0.400\,, & \tilde{x} = \tilde{x}_{II} \\ 0.217\,, & \tilde{x} = \tilde{x}_{III}\,, \end{cases}$$

$$\tilde{f}_1(\tilde{x}) = \int_0^1 H_1^*(\theta)\tilde{f}(\tilde{x};\theta)\, d\theta = \begin{cases} 0.217\,, & \tilde{x} = \tilde{x}_I \\ 0.400\,, & \tilde{x} = \tilde{x}_{II} \\ 0.383\,, & \tilde{x} = \tilde{x}_{III}\,. \end{cases}$$

Therefore, we get

$$z_i = \log \frac{\tilde{f}_0(\tilde{x}_i)}{\tilde{f}_1(\tilde{x}_i)} = \begin{cases} 0.570\,, & \tilde{x}_i = \tilde{x}_I \\ 0.000\,, & \tilde{x}_i = \tilde{x}_{II} \\ -0.570\,, & \tilde{x}_i = \tilde{x}_{III}\,. \end{cases}$$

Let $\alpha_{\tilde{\Phi}} = 0.05$ and $\beta_{\tilde{\Phi}} = 0.01$. We obtain $\log(k_0') = -2.9857$, $\log(k_1') = 4.5539$, $\mathrm{E}[Z_i|H_0 \text{ true}] = 0.095$, and $\mathrm{E}[Z_i|H_1 \text{ true}] = -0.095$. Hence, $\mathrm{E}[N|H_0 \text{ true}] = 43.968$, and we must take $n = 44$, whereas $\mathrm{E}[N|H_1 \text{ true}] = 30.635$, thus we take $n = 31$.

Acknowledgement

The authors are grateful to the referees of the journal for their suggestions and for their help in writing the paper in acceptable form.

References

Arnold, B. F. (1995). Statistical tests optimally meeting certain fuzzy requirements on the power function and on the sample size. *Fuzzy Sets and Systems*, *75*, 365-372.

Arnold, B. F. (1996). An approach to fuzzy hypotheses testing. *Metrika*, *44*, 119-126.

Arnold, B. F. (1998). Testing fuzzy hypothesis with crisp data. *Fuzzy Sets and Systems*, *94*, 323-333.

Ash, R. B., and Doleans-Dade, C. A. (2000). *Probability and Measure Theory* (2nd ed.). San Diego: Academic Press.

Billingsley, P. (1995). *Probability and Measure* (2nd ed.). New York: John Wiley.

Casals, M. R., Gil, M. R., and Gil, P. (1986). On the use of Zadeh's probabilistic definition for testing statistical hypotheses from fuzzy information. *Fuzzy Sets and Systems*, *20*, 175-190.

Casella, G., and Berger, R. L. (2002). *Statistical Inference* (2nd ed.). New York: Duxbury Press.

Chung, K. L. (2000). *A Course in Probability Theory* (2nd ed.). New York: Academic Press.

Delgado, M., Verdegay, J. L., and Vila, M. A. (1985). Testing fuzzy hypotheses: A Bayesian approach. In M. M. G. et al. (Ed.), *Approximate Reasoning in Expert Systems* (p. 307-316). Amsterdam: North-Holland.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications* (3rd ed., Vol. 1). New York: John Wiley.

Grzegorzewski, P. (2000). Testing statistical hypotheses with vague data. *Fuzzy Sets and Systems*, *112*, 501-510.

Grzegorzewski, P. (2002). Testing fuzzy hypotheses with vague data. In C. Bertoluzzi (Ed.), *Statistical Modelling, Analysis and Management of Fuzzy Data* (p. 213-225). Heidelberg: Physica.

Hogg, R. V., and Craig, A. T. (1995). *An Introduction to Mathematical Statistics* (5th ed.). Eaglewood Cliffs: Prentice-Hall.

Juninig, O., and Wang, G. (1989). Fuzzy random variables and their probability characters. *Journal of Harbin Architectural and Civil Engineering Institute*, *12*, 1-10.

Kruse, R., and Meyer, K. D. (1987). *Statistics with Vague Data*. Dordrecht: D. Reidel Publishing Company.

Kwakernaak, H. (1978). Fuzzy random variables - I: Definitions and theorems. *Information Science*, *15*, 1-29.

Lehmann, E. L. (1994). *Testing Statistical Hypotheses*. London: Chapman-Hall.

Liu, B. (2004). *Uncertainty Theory: An Introduction to its Axiomatic Foundation*. Heidelberg: Physica.

López-Díaz, M., and Gil, M. A. (1997). Constructive definitions of fuzzy random variables. *Statistics & Probability Letters*, *36*, 135-144.

López-Díaz, M., and Gil, M. A. (1998). Approximately integrably bounded fuzzy random variable, in terms of the "generalized" Hausdorff metric. *Information Science*, *104*, 279-291.

Mood, A. M., Graybill, F. A., and Boes, D. C. (1974). *Introduction to the Theory of Statistics* (3rd ed.). London: McGraw-Hill.

Puri, M. L., and Ralescu, D. A. (1986). Fuzzy random variables. *Journal of Mathematical Analysis and Applications*, *114*, 409-422.

Ralescu, D. (1995). Fuzzy random variable revisited. In *Proc. ifes' 95* (Vol. 2, p. 993-1000).

Ross, S. M. (2002). *A First Course in Probability* (6th ed.). New Jersey: Prentice-Hall.

Saade, J. (1994). Extension of fuzzy hypotheses testing with hybrid data. *Fuzzy Sets and Systems*, *63*, 57-71.

Saade, J., and Schwarzlander, H. (1990). Fuzzy hypotheses testing with hybrid data. *Fuzzy Sets and Systems*, *35*, 192-212.

Shao, J. (1999). *Mathematical Statistics*. New York: Springer.

Son, J. C., Song, I., and Kim, H. Y. (1992). A fuzzy decision problem based on the generalized Neyman-Pearson criteria. *Fuzzy Sets and Systems*, *47*, 65-75.

Taheri, S. M. (2003). Trends in fuzzy statistics. *Austrian Journal of Statistics*, *32*, 239-257.

Taheri, S. M., and Behboodian, J. (1999). Neyman-Pearson lemma for fuzzy hypotheses testing. *Metrika*, *49*, 3-17.

Taheri, S. M., and Behboodian, J. (2001). A Bayesian approach to fuzzy hypotheses testing. *Fuzzy Sets and Systems*, *123*, 39-48.

Taheri, S. M., and Behboodian, J. (2002). Fuzzy hypotheses testing with fuzzy data: A Bayesian approach. In R. P. Nikhil and M. Sugeno (Eds.), *Advances in soft computing, afss 2002* (p. 527-533).

Watanabe, N., and Imaizumi, T. (1993). A fuzzy statistical test of fuzzy hypotheses. *Fuzzy Sets and Systems*, *53*, 167-178.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338-353.

Zadeh, L. A. (1968). Probability measures of fuzzy events. *Journal of Mathematical Analysis and Applications*, *23*, 421-427.

Author's addresses:

Hamzeh Torabi and Javad Behboodian
Department of Statistics
Shiraz University
Shiraz, Iran

E-mail: htorabi@yazduni.ac.ir