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On the Product and Ratio of Pearson Type VII and Laplace Random Variables

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Abstract: The distributions of the product $|XY|$ and the ratio $|X/Y|$ are derived when X and Y are Pearson type VII and Laplace random variables distributed independently of each other. Extensive tabulations of the associated percentage points are also given.

Zusammenfassung: Die Verteilungen des Produkts $|XY|$ und des Quotienten $|X/Y|$ sind hergeleitet, falls X und Y unabhängig verteilte Pearson type VII und Laplace Zufallsvariablen sind. Umfangreiche Tabellen der dazugehörenden Perzentile werden auch angegeben.

Keywords:

1 Introduction

For given random variables X and Y, the distributions of the product $|XY|$ and the ratio $|X/Y|$ are of interest in many areas of the sciences, engineering and medicine. Examples of $|XY|$ include traditional portfolio selection models (Grubel, 1968), relationship between attitudes and behavior (Rokeach and Kliejunas, 1972), number of cancer cells in tumor biology (Ladekarl et al., 1997) and stream flow in hydrology (Cigizoglu and Bayazit, 2000). Examples of $|X/Y|$ include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and safety factor in engineering systems (see, for example, Kotz et al., 2003).

The distributions of $|XY|$ and $|X/Y|$ have been studied by several authors especially when X and Y are independent random variables and come from the same family. With respect to $|XY|$, see Sakamoto (1943) for the uniform family, Harter (1951) and Wallgren (1980) for the Student's t family, Springer and Thompson (1970) for the normal family, Stuart (1962) and Podolski (1972) for the gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for the beta family, Abu-Salih (1983) for the power function family, and Malik and Trudel (1986) for the exponential family (see also Rathie and Rohrer, 1987, for a comprehensive review of known results). With respect to $|X/Y|$, see Marsaglia (1965) and Korhonen and Narula (1989) for the normal family, Press (1969) for the Student's t family, Basu and Lochner (1971) for the Weibull family, Shcolnick (1985) for the stable family, Hawkins and Han (1986) for the non-central chi-squared family, Provost (1989) for the gamma family, and Pham-Gia (2000) for the beta family.

However, there is relatively little work of this kind when X and Y belong to different families. In the applications mentioned above, it is quite possible that X and Y could arise from different but similar distributions. In this note, we study the distributions of $|XY|$ and $|X/Y|$ when X and Y are independent random variables having the Pearson type VII and Laplace distributions with pdfs

$$f(x) = \frac{\Gamma(M-1/2)}{\sqrt{m\pi}\Gamma(M-1)} \left(1+\frac{x^2}{m}\right)^{1/2-M} \tag{1}$$

and

$$f(y) = \frac{\lambda}{2} \exp\left(-\lambda |y|\right), \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $m > 0$, $M > 1$, and $\lambda > 0$. Extensive tabulations of the associated percentage points are also provided. Since λ is just a scale parameter we shall assume without loss generality that $\lambda = 1$.

The results of this note use the following relationship between the Pearson type VII distribution and the well-known Student's t distribution: if $M = 1 + a/2$ and

$$U = \sqrt{\frac{a}{m}} X \tag{3}$$

then U is a Student's t random variable with a degrees of freedom. Note that the pdf of a Student's t random variable with degrees of freedom ν is given by

$$f(x) = \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \left(1+\frac{x^2}{\nu}\right)^{-(1+\nu)/2} \tag{4}$$

for $-\infty < x < \infty$. Nadarajah and Kotz (2003) have shown that the cdf corresponding to (4) can be expressed as

$$F(x) = \begin{cases} \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{\sqrt{\nu}}\right) + \frac{1}{2\pi} \sum_{l=1}^{(\nu-1)/2} B\left(l, \frac{1}{2}\right) \frac{\nu^{l-1/2} x}{\left(\nu + x^2\right)^{l}} & \text{if } \nu \text{ is odd,} \\ \frac{1}{2} + \frac{1}{2\pi} \sum_{l=1}^{\nu/2} B\left(l - \frac{1}{2}, \frac{1}{2}\right) \frac{\nu^{l-1} x}{\left(\nu + x^2\right)^{l-1/2}} & \text{if } \nu \text{ is even.} \end{cases} \tag{5}$$

This result will be crucial for the calculations of this note. The calculations involve the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{x^{-t} \Gamma\left(b_1 + t\right) \cdots \Gamma\left(b_m + t\right) \Gamma\left(1 - a_1 - t\right) \cdots \Gamma\left(1 - a_n - t\right)}{\Gamma\left(a_{n+1} + t\right) \cdots \Gamma\left(a_p + t\right) \Gamma\left(1 - b_{m+1} - t\right) \cdots \Gamma\left(1 - b_q - t\right)} dt,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial and L denotes an integration path defined as one of:

- the path L running from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(1-a_k+t)$ lie to the left, and the poles of the functions $\Gamma(b_j - t)$ lie to the right of L for $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

- a loop, beginning and ending at $+\infty$, that encircles the poles of the functions $\Gamma(b_j - t)$ for $j = 1, \ldots, m$ once in the negative direction. All the poles of the functions $\Gamma(1 - a_k + t)$ must remain outside this loop.
- a loop L, beginning and ending at $-\infty$, that encircles the poles of the functions $\Gamma(1 - a_k + t)$ for $k = 1, \ldots, n$ once in the positive direction. All the poles of the functions $\Gamma(b_j - t)$ for $j = 1, \ldots, m$ must remain outside this loop.

In all three cases, the poles of $\Gamma(b_j - t)$ must not coincide with the poles of $\Gamma(1 - a_k + t)$ for any $j = 1, \ldots, m$ and $k = 1, \ldots, n$ (see Section 9.3 in Gradshteyn and Ryzhik, 2000, for further details).

The calculations of this note also need the following important lemma.

Lemma 1 *(Equation (3.389.2), Gradshteyn and Ryzhik, 2000) For $\mu > 0$ and $\nu > 0$,*

$$\int_0^\infty x^{2\nu-1} \left(u^2 + x^2\right)^{\rho-1} \exp(-\mu x) dx = \frac{u^{2\nu+2\rho-2}}{2\sqrt{\pi}\Gamma(1-\rho)} G_{13}^{31} \left(\frac{\mu^2 u^2}{4} \middle| \begin{array}{c} 1-\nu \\ 1-\rho-\nu, 0, \frac{1}{2} \end{array} \right).$$

Further properties of the Meijer G-function can be found in Erdelyi et al. (1981), Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 Product

The cdf of the Student's t distribution takes different forms depending on whether its degrees of freedom parameter is an odd integer or even integer (see Nadarajah and Kotz, 2003, and references therein). Note that in (3) the degrees of freedom parameter $a = 2(M-1)$. Thus, one would expect the cdf of $|XY|$ to be different depending on whether a is an odd integer or not. Theorems 1 and 2 derive explicit expressions for the cdf of $|XY|$ for these two cases.

Theorem 1 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. If $a = 2(M-1)$ is an odd integer then the cdf of $Z = |XY|$ can be expressed as*

$$F(z) = I(a) + \frac{r}{2\pi\sqrt{a}} \sum_{k=1}^{(a-1)/2} \frac{1}{\Gamma(k+1/2)} G_{13}^{31} \left(\frac{r^2}{4a} \middle| \begin{array}{c} 1-k \\ 0, 0, \frac{1}{2} \end{array} \right), \tag{6}$$

where $r = \sqrt{a/m}z$ and $I(\cdot)$ denotes the integral

$$I(a) = \frac{2}{\pi} \int_0^\infty \arctan\left(\frac{r}{\sqrt{ay}}\right) \exp(-y)\, dy. \tag{7}$$

Proof: Using the relationship (3), one can write the cdf as $\Pr(|XY| \leq z) = \Pr(|UY| \leq r)$, which can be expressed as

$$\begin{aligned} F(r) &= \frac{1}{2} \int_{-\infty}^\infty \left\{ F\left(\frac{r}{|y|}\right) - F\left(-\frac{r}{|y|}\right) \right\} \exp(-|y|)\, dy \\ &= \int_0^\infty \left\{ F\left(\frac{r}{y}\right) - F\left(-\frac{r}{y}\right) \right\} \exp(-y)\, dy, \end{aligned} \tag{8}$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's t random variable with degrees of freedom a. Substituting the form for F given by (5) for odd degrees of freedom, (8) can be reduced to

$$F(r) = I(a) + \frac{r}{\pi\sqrt{a}} \sum_{k=1}^{(a-1)/2} B\left(k, \frac{1}{2}\right) J(k), \tag{9}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{2k-1}\exp(-y)}{\left(y^2 + r^2/a\right)^k} dy. \tag{10}$$

By direct application of Lemma 1, one can calculate (10) as

$$J(k) = \frac{1}{2\sqrt{\pi}\Gamma(k)} G_{13}^{31}\left(\frac{r^2}{4a}\middle| \begin{matrix} 1-k \\ 0, 0, \frac{1}{2} \end{matrix}\right). \tag{11}$$

The result of the theorem follows by substituting (11) into (9). ■

Theorem 2 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. If $a = 2(M-1)$ is an even integer then the cdf of $Z = |XY|$ can be expressed as*

$$F(z) = \frac{r}{2\pi\sqrt{a}} \sum_{k=1}^{a/2} \frac{1}{\Gamma(k)} G_{13}^{31}\left(\frac{r^2}{4a}\middle| \begin{matrix} \frac{3}{2}-k \\ 0, 0, \frac{1}{2} \end{matrix}\right), \tag{12}$$

where $r = \sqrt{a/m}z$.
Proof: Substituting the form for F given by (5) for even degrees of freedom, (8) can be reduced to

$$F(z) = \frac{r}{\pi\sqrt{a}} \sum_{k=1}^{a/2} B\left(k - \frac{1}{2}, \frac{1}{2}\right) J(k), \tag{13}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{2k-2}\exp(-y)}{\left(y^2 + r^2/a\right)^{k-1/2}} dy. \tag{14}$$

By direct application of Lemma 1, one can calculate (14) as

$$J(k) = \frac{1}{2\sqrt{\pi}\Gamma(k-1/2)} G_{13}^{31}\left(\frac{r^2}{4a}\middle| \begin{matrix} \frac{3}{2}-k \\ 0, 0, \frac{1}{2} \end{matrix}\right). \tag{15}$$

The result of the theorem follows by substituting (15) into (13). ■

Note that the parameters in (6) and (12) are functions of $\sqrt{M-1}$. Figure 1 illustrates possible shapes of the pdf of $|XY|$ for a range of values of $\sqrt{M-1}$. Note that the shapes are unimodal and that the value of $\sqrt{M-1}$ largely dictates the behavior of the pdf near $z = 0$.

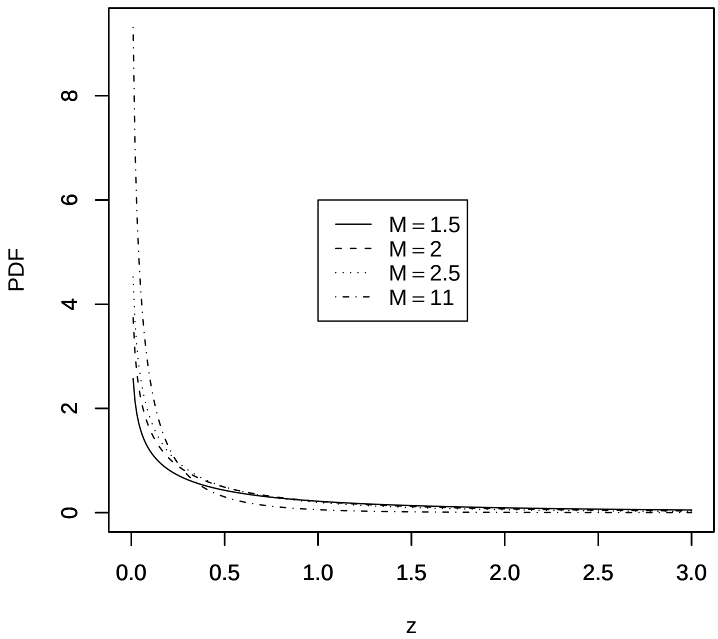

Figure 1: Plots of the pdf of (6) and (12) for $m = 1$ and $M = 1.5, 2, 2.5, 11$.

3 Ratio

For the reasons mentioned in Section 2, the cdf of $|X/Y|$ will be different depending on whether a is an odd integer or not. Theorems 3 and 4 derive explicit expressions for the cdf of $|X/Y|$ for these two cases.

Theorem 3 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. If $a = 2(M-1)$ is an odd integer then the cdf of $Z = |X/Y|$ can be expressed as*

$$F(z) = I(a) + \frac{\sqrt{a}}{2\pi r} \sum_{k=1}^{(a-1)/2} \frac{1}{\Gamma(k+1/2)} G_{13}^{31}\left(\frac{a}{4r^2} \middle| \begin{matrix} 0 \\ k-1, 0, \frac{1}{2} \end{matrix} \right), \tag{16}$$

where $r = \sqrt{a/mz}$ and $I(\cdot)$ denotes the integral

$$I(a) = \frac{2}{\pi} \int_0^\infty \arctan\left(\frac{ry}{\sqrt{a}}\right) \exp(-y)\, dy. \tag{17}$$

Proof: Using the relationship (3), one can write the cdf as $\Pr(|X/Y| \leq z) = \Pr(|U/Y| \leq r)$, which can be expressed as

$$\begin{aligned} F(r) &= \frac{1}{2}\int_{-\infty}^{\infty} \{F(r|y|) - F(-r|y|)\} \exp(-|y|)\, dy \\ &= \int_0^\infty \{F(ry) - F(-ry)\} \exp(-y)\, dy, \end{aligned} \tag{18}$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's t random variable with degrees of freedom a. Substituting the form for F given by (5) for odd degrees of freedom, (18) can be reduced to

$$F(r) = I(a) + \frac{r}{\pi\sqrt{a}} \sum_{k=1}^{(a-1)/2} a^k B\left(k, \frac{1}{2}\right) J(k), \tag{19}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y \exp(-y)}{\left(y^2 r^2 + a\right)^k} dy. \tag{20}$$

By direct application of Lemma 1, one can calculate (20) as

$$J(k) = \frac{a^{1-k}}{2\sqrt{\pi}\Gamma(k) r^2} G_{13}^{31}\left(\frac{a}{4r^2}\middle| \begin{matrix} 0 \\ k-1, 0, \frac{1}{2} \end{matrix}\right). \tag{21}$$

The result of the theorem follows by substituting (21) into (19). ■

Theorem 4 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. If $a = 2(M-1)$ is an even integer then the cdf of $Z = |X/Y|$ can be expressed as*

$$F(z) = \frac{\sqrt{a}}{2\pi r} \sum_{k=1}^{a/2} \frac{1}{\Gamma(k)} G_{13}^{31}\left(\frac{a}{4r^2}\middle| \begin{matrix} 0 \\ k-\frac{3}{2}, 0, \frac{1}{2} \end{matrix}\right), \tag{22}$$

where $r = \sqrt{a/m}z$.

Proof: Substituting the form for F given by (5) for even degrees of freedom, (18) can be reduced to

$$F(r) = \frac{r}{\pi a} \sum_{k=1}^{a/2} a^k B\left(k - \frac{1}{2}, \frac{1}{2}\right) J(k), \tag{23}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y \exp(-y)}{\left(y^2 r^2 + a\right)^{k-1/2}} dy. \tag{24}$$

By direct application of Lemma 1, one can calculate (24) as

$$J(k) = \frac{a^{3/2-k}}{2\sqrt{\pi}\Gamma(k-1/2) r^2} G_{13}^{31}\left(\frac{a}{4r^2}\middle| \begin{matrix} 0 \\ k-\frac{3}{2}, 0, \frac{1}{2} \end{matrix}\right). \tag{25}$$

The result of the theorem follows by substituting (25) into (23). ■

Note that the parameters in (16) and (22) are functions of $\sqrt{M-1}$. Figure 2 illustrates possible shapes of the pdf of $|X/Y|$ for a range of values of $\sqrt{M-1}$. Note that the shapes are unimodal and that the value of $\sqrt{M-1}$ largely dictates the behavior of the pdf near $z = 0$.

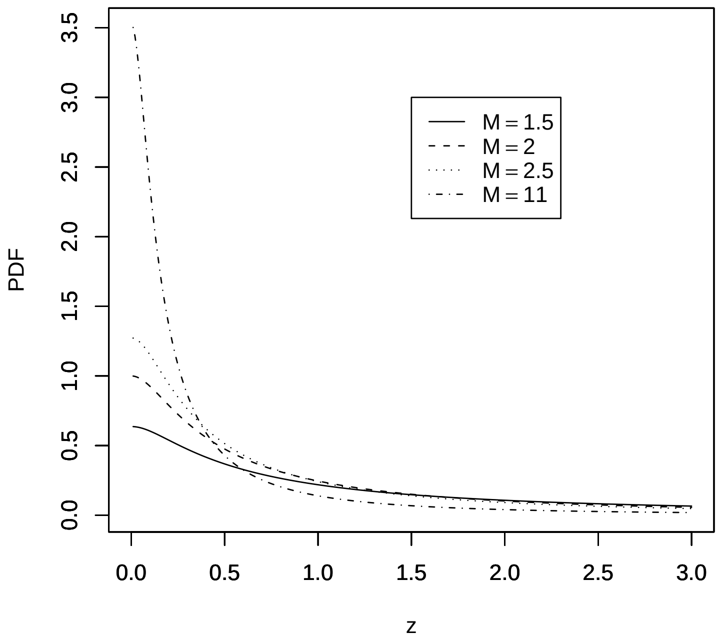

Figure 2: Plots of the pdf of (16) and (22) for $m = 1$ and $M = 1.5, 2, 2.5, 11$.

4 Percentiles

Almost every textbook in statistics has tables of percentage points of the Student's t distribution for integer values of its degrees of freedom parameter. In this section, we provide tabulations of percentage points z_p of $|XY|$ and $|X/Y|$ for integer values of the degrees of freedom parameter $a = 2(M-1)$; see equation (3). We feel tables (and not pictures) are the right way to present percentage points. Of course, it is of no use to have pictures of percentile points if one needs to use them.

We have used the results in Theorems 1 to 4 to compute the percentile points. The percentage points z_p of $|XY|$ are obtained by numerically solving the equations

$$I(a) + \frac{r}{2\pi\sqrt{a}} \sum_{k=1}^{(a-1)/2} \frac{1}{\Gamma(k+1/2)} G_{13}^{31}\left(\frac{r^2}{4a} \,\middle|\, \begin{matrix} 1-k \\ 0, 0, \frac{1}{2} \end{matrix} \right) = p$$

and

$$\frac{r}{2\pi\sqrt{a}} \sum_{k=1}^{a/2} \frac{1}{\Gamma(k)} G_{13}^{31}\left(\frac{r^2}{4a} \,\middle|\, \begin{matrix} \frac{3}{2}-k \\ 0, 0, \frac{1}{2} \end{matrix} \right) = p,$$

where $r = \sqrt{a/m} z_p$, $a = 2(M-1)$ and $I(a)$ is given by the integral in (7). The percentage points z_p of $|X/Y|$ are obtained by numerically solving the equations

$$I(a) + \frac{\sqrt{a}}{2\pi r} \sum_{k=1}^{(a-1)/2} \frac{1}{\Gamma(k+1/2)} G_{13}^{31}\left(\frac{a}{4r^2} \,\middle|\, \begin{matrix} 0 \\ k-1, 0, \frac{1}{2} \end{matrix} \right) = p$$

and

$$\frac{\sqrt{a}}{2\pi r}\sum_{k=1}^{a/2}\frac{1}{\Gamma(k)}G_{13}^{31}\left(\frac{a}{4r^2}\,\middle|\,\begin{matrix}0\\ k-\frac{3}{2},0,\frac{1}{2}\end{matrix}\right)=p,$$

where $a = 2(M-1)$, $r = \sqrt{a/m}z_p$, and $I(a)$ is given by the integral in (17). Evidently, this involves computation of the Meijer G function and routines for this are widely available. We used the function MeijerG $(\cdot)$ in the algebraic manipulation package MAPLE. The tables below provide the numerical values of z_p for $m = 1$ and $2(M-1) = 1, \dots, 50$.

Percentage points z_p of $Z = \mid XY \mid$.

$2(M{-}1)$	$p=0.01$	$p=0.05$	$p=0.1$	$p=0.9$	$p=0.95$	$p=0.99$
1	0.002472114	0.01760725	0.04390443	6.118916	12.58945	63.91723
2	0.001514041	0.01075407	0.02661374	2.298551	3.765078	9.623113
3	0.001195504	0.008314026	0.02058396	1.577600	2.452657	5.400213
4	0.001000614	0.007021931	0.01720126	1.260786	1.921103	3.98975
5	0.0008797462	0.006266828	0.01530928	1.075392	1.623685	3.287362
6	0.000789633	0.005582115	0.01365501	0.9574891	1.433776	2.857114
7	0.0007313845	0.005114393	0.01256056	0.8680605	1.291771	2.525688
8	0.0006815406	0.004803871	0.01176462	0.7993727	1.191321	2.306157
9	0.0006481296	0.004483806	0.01092727	0.7440212	1.107616	2.143675
10	0.0006029902	0.004277564	0.01046241	0.7010453	1.037201	2.004647
11	0.0005667269	0.004029889	0.009888954	0.6646361	0.983732	1.889267
12	0.0005453437	0.003824626	0.009374375	0.6293656	0.9320625	1.791484
13	0.0005231168	0.003674582	0.00903843	0.6039542	0.891087	1.713002
14	0.0005049957	0.003542788	0.008657438	0.5795731	0.8534818	1.625002
15	0.0004898986	0.003409776	0.008396631	0.558168	0.8215026	1.574154
16	0.0004746198	0.003292211	0.008097579	0.5376444	0.792269	1.511334
17	0.0004564091	0.003204449	0.007851023	0.5193	0.7640655	1.461276
18	0.0004398412	0.003091821	0.007587078	0.5037341	0.7414135	1.415446
19	0.0004252827	0.003009028	0.007377427	0.4882129	0.72024	1.368343
20	0.0004282975	0.002972889	0.007241078	0.4783454	0.7047234	1.336702
21	0.0004082291	0.002875724	0.007033738	0.4640536	0.6811964	1.290705
22	0.000404547	0.002810075	0.006872857	0.4521249	0.6639954	1.259963
23	0.0003906085	0.002745529	0.006679763	0.4409385	0.6477942	1.224096
24	0.0003835892	0.002698069	0.006619145	0.4327369	0.6353156	1.197652
25	0.0003822950	0.002647954	0.00646853	0.4227546	0.6219806	1.176744
26	0.0003766123	0.002589539	0.00630544	0.4138431	0.6078777	1.147442
27	0.0003560115	0.002518562	0.006170799	0.4067685	0.5984383	1.129445
28	0.000357231	0.002485029	0.006045279	0.3987274	0.5852882	1.110611
29	0.0003433643	0.002426284	0.005966482	0.3897082	0.5722598	1.088156
30	0.0003391172	0.002388773	0.005833811	0.3842614	0.5644508	1.070804
31	0.0003327494	0.002338561	0.005750443	0.3768030	0.554324	1.045861
32	0.0003290055	0.002311220	0.00564555	0.3715522	0.543171	1.027811
33	0.0003257495	0.002272163	0.005592716	0.3661364	0.5366617	1.009437
34	0.0003164533	0.002240309	0.00551071	0.3590140	0.5273947	0.993899
35	0.0003130629	0.002223908	0.005431096	0.3534908	0.5198634	0.979372

36	0.0003055219	0.002174063	0.005337687	0.3492768	0.5125242	0.9648715
37	0.0002990636	0.00213066	0.005255926	0.3427273	0.5019898	0.943434
38	0.0003056362	0.002132106	0.005206365	0.339356	0.497437	0.939157
39	0.0002994669	0.002084988	0.005085461	0.3343967	0.4912639	0.9246274
40	0.0002943588	0.002059693	0.005055784	0.3301844	0.4836621	0.9072937
41	0.0002883626	0.002034850	0.005015785	0.3257089	0.4778085	0.89973
42	0.0002806783	0.002015011	0.004930961	0.3218926	0.4720065	0.8907853
43	0.0002838914	0.001987437	0.004851522	0.318053	0.4658031	0.8779472
44	0.000283784	0.001977548	0.00485642	0.3147179	0.4608815	0.8676147
45	0.0002811036	0.00195936	0.004786425	0.3109548	0.4562908	0.859963
46	0.0002832859	0.001930131	0.00472959	0.3077832	0.4510431	0.8519999
47	0.0002724619	0.001899337	0.004660093	0.3035679	0.445435	0.8388599
48	0.0002725358	0.001895982	0.004631664	0.3006243	0.4404166	0.826102
49	0.0002638080	0.001856912	0.004560978	0.2972973	0.4348552	0.8142045
50	0.0002594850	0.001836983	0.00450004	0.293448	0.4314492	0.8125693

Percentage points z_p of $Z = |X/Y|$.

$2(M-1)$	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
1	0.01552512	0.0790026	0.1641902	22.88558	56.84977	411.0098
2	0.009975643	0.05015735	0.1026342	8.28857	18.00793	96.2985
3	0.007816286	0.03968661	0.08057413	5.668537	12.03303	61.78769
4	0.006626128	0.03354873	0.0679197	4.512941	9.491855	49.42577
5	0.005887559	0.02952082	0.06005404	3.85519	8.072666	42.21947
6	0.005408283	0.0271436	0.05481008	3.432993	7.19464	37.64227
7	0.004980384	0.02473773	0.05016065	3.09359	6.448568	33.39787
8	0.004530498	0.02298499	0.04645628	2.854718	6.007938	30.91027
9	0.004308939	0.02151251	0.04373956	2.656687	5.564783	28.63196
10	0.004095285	0.02043785	0.04140724	2.50089	5.238181	27.16365
11	0.003884962	0.01945167	0.03932117	2.380731	4.963346	25.9658
12	0.003700266	0.01855726	0.03763133	2.245607	4.695622	23.78158
13	0.003567936	0.01785324	0.03600228	2.15192	4.474087	22.87406
14	0.003405106	0.01712076	0.03467307	2.065610	4.317228	22.17280
15	0.003267813	0.0164992	0.03340208	1.994268	4.1557	21.35417
16	0.003146753	0.01595784	0.0322582	1.923907	4.006011	20.68539
17	0.003088841	0.01542873	0.03124269	1.858214	3.895033	20.28069
18	0.003005947	0.01505406	0.03051371	1.798431	3.753345	19.53511
19	0.002901368	0.01463949	0.02974099	1.755702	3.664237	19.05411
20	0.002829482	0.01434412	0.02894624	1.698965	3.554642	18.22927
21	0.002770815	0.01383819	0.02809195	1.662537	3.472463	18.09284
22	0.002723278	0.01355855	0.02750494	1.631907	3.422312	17.60027
23	0.002667917	0.01331303	0.02684050	1.582342	3.310638	17.17738
24	0.002571521	0.01303875	0.02626474	1.545490	3.235744	16.80679
25	0.002549789	0.01270856	0.02567412	1.517908	3.186356	16.38278
26	0.00248588	0.01250269	0.02528630	1.479835	3.085666	16.05842
27	0.002424534	0.01226024	0.02484678	1.453603	3.033881	15.52877
28	0.002403672	0.01203829	0.02441463	1.421769	2.963977	15.25569

29	0.002349949	0.01179345	0.02380616	1.397971	2.922000	14.92001
30	0.002268749	0.01150689	0.02338628	1.366414	2.82414	14.48084
31	0.002261365	0.01138858	0.02301815	1.345269	2.812673	14.54822
32	0.002222787	0.01123649	0.02262378	1.333248	2.789840	14.40294
33	0.002191962	0.01102161	0.02237726	1.307092	2.743965	14.13491
34	0.002156419	0.01085153	0.02205565	1.284272	2.688816	13.98358
35	0.002129226	0.01066998	0.02167717	1.266089	2.645041	13.61800
36	0.002117382	0.01047937	0.02132661	1.243532	2.603189	13.52347
37	0.002089759	0.01041228	0.02099816	1.225085	2.557589	13.39218
38	0.002054058	0.01030938	0.02089264	1.215141	2.556463	13.26175
39	0.002025260	0.01013767	0.02057693	1.204689	2.49853	13.01250
40	0.001999654	0.01000962	0.02027367	1.178909	2.455118	12.65659
41	0.001950654	0.009836828	0.01998355	1.170412	2.443197	12.41125
42	0.001959500	0.009812757	0.01985281	1.145491	2.379911	12.30821
43	0.001922644	0.009670885	0.01952248	1.137936	2.367750	12.29584
44	0.001894266	0.00952382	0.01931153	1.132921	2.37106	12.22001
45	0.001882979	0.009456378	0.0190996	1.113959	2.314872	11.90647
46	0.00186541	0.009381124	0.01896228	1.099207	2.303544	11.71348
47	0.001857872	0.009298325	0.01877732	1.090312	2.275117	11.76624
48	0.001803032	0.009084494	0.01846785	1.077119	2.244608	11.53742
49	0.001786668	0.009032666	0.0183073	1.064223	2.223506	11.62009
50	0.001767938	0.008907266	0.01801453	1.056736	2.204715	11.37300

We hope these numbers will be of use to the practitioners mentioned in Section 1. Similar tabulations could be easily derived for other values of $2(M-1)$ by using the MeijerG $(\cdot)$ function in MAPLE. Sample programs are shown in the Appendix.

Appendix

The following programs in MAPLE can be used to generate tables similar to those presented in Section 4.

```
#this program gives percentiles of |XY| when df 2*(M-1) is odd
a:=2*(M-1):
r:=sqrt(a/m)*z:
ff:=(2/Pi)*int(arctan(r/(sqrt(a)*y))*exp(-y),y=0..infinity):
for k from 1 to (a-1)/2 do
ttt:=MeijerG([[1-k],[]],[[0,0,1/2],[]],(r**2)/(4*a)):
ff:=ff+ttt*r/(2*sqrt(a)*Pi*GAMMA(k+1/2)):
od;
p1:=fsolve(ff=0.01,z=0..10000):
p2:=fsolve(ff=0.05,z=0..10000):
p3:=fsolve(ff=0.1,z=0..10000):
p4:=fsolve(ff=0.90,z=0..10000):
```

```
p5:=fsolve(ff=0.95,z=0..10000):
p6:=fsolve(ff=0.99,z=0..10000):
print(a,p1,p2,p3,p4,p5,p6);

#this program gives percentiles of |XY| when df 2*(M-1) is even
a:=2*(M-1): r:=sqrt(a/m)*z: ff:=0: for k from 1 to a/2 do
ttt:=MeijerG([[3/2-k],[]],[[0,0,1/2],[]],(r**2)/(4*a)):
ff:=ff+ttt*r/(2*sqrt(a)*Pi*GAMMA(k)): od;
p1:=fsolve(ff=0.01,z=0..10000): p2:=fsolve(ff=0.05,z=0..10000):
p3:=fsolve(ff=0.1,z=0..10000): p4:=fsolve(ff=0.90,z=0..10000):
p5:=fsolve(ff=0.95,z=0..10000): p6:=fsolve(ff=0.99,z=0..10000):
print(a,p1,p2,p3,p4,p5,p6);

#this program gives percentiles of |X/Y| when df 2*(M-1) is odd
a:=2*(M-1):
r:=sqrt(a/m)*z:
ff:=(2/Pi)*int(arctan(r*y/sqrt(a))*exp(-y),y=0..infinity):
for k from 1 to (a-1)/2 do
ttt:=MeijerG([[0],[]],[[k-1,0,1/2],[]],a/(4*r*r)):
ff:=ff+ttt*sqrt(a)/(2*r*Pi*GAMMA(k+1/2)):
od;
p1:=fsolve(ff=0.01,z=0..10000):
p2:=fsolve(ff=0.05,z=0..10000):
p3:=fsolve(ff=0.1,z=0..10000):
p4:=fsolve(ff=0.90,z=0..10000):
p5:=fsolve(ff=0.95,z=0..10000):
p6:=fsolve(ff=0.99,z=0..10000):
print(a,p1,p2,p3,p4,p5,p6);

#this program gives percentiles of |X/Y| when df 2*(M-1) is even
a:=2*(M-1):
r:=sqrt(a/m)*z:
ff:=0:
for k from 1 to a/2 do
ttt:=MeijerG([[0],[]],[[k-3/2,0,1/2],[]],a/(4*r*r)):
ff:=ff+ttt*sqrt(a)/(2*r*Pi*GAMMA(k)):
od;
p1:=fsolve(ff=0.01,z=0..10000):
p2:=fsolve(ff=0.05,z=0..10000):
p3:=fsolve(ff=0.1,z=0..10000):
p4:=fsolve(ff=0.90,z=0..10000):
p5:=fsolve(ff=0.95,z=0..10000):
p6:=fsolve(ff=0.99,z=0..10000):
print(a,p1,p2,p3,p4,p5,p6);
```

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