Stationary processes and a one sided representation in terms of independent identically distributed random variables.

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Abstract

N. Wiener conjectured a necessary and sufficient condition for a stationary process to have a one sided representation (causal and possibly nonlinear) in terms of independent identically distributed random variables—that the process be purely nondeterministic (infinite past trivial). The sub-domains in which the conjecture holds or fails are discussed.

This article is dedicated to the memory of Walter Philipp who contributed so much to the field of stochastics. I also dedicate this paper to my dear wife Ady who sadly died this June 4, 2009 at the inpatient unit of the San Diego Hospice.

Introduction

Consider a stationary sequence $\{X_n, -\infty < n < \infty\}$ with

$$\mathcal{B}_n = \mathcal{B}\{X_j, j \leq n\} \tag{1}$$

the σ-field generated by $X_j, j \leq n$. The sequence $\{\xi_n, -\infty < n < \infty\}$ is a sequence of independent, identically distributed random variables. In Wiener (1958) under what circumstances a stationary process $\{x_n\}$ could have a one-sided representation

$$x_n = f(\xi_n, \xi_{n-1}, \dots) \tag{2}$$

in terms of an iid sequence was considered. Wiener conjectured that a necessary and sufficient condition for such a representation was that the backward tail field

$$\mathcal{B}_{-\infty} = \cap_n \mathcal{B}_n = \{\emptyset, \Omega\}$$

be trivial. Here $\emptyset$ is the empty set and Ω the whole space. In Rosenblatt (1960) this was shown to be true for countable state Markov chains. An extension for a class of continuous state Markov sequences was given in Hanson (1963). Hanson's condition was that $\{X_n\}$ is a real-valued stationary Markov sequence with (i) trivial tail field $\mathcal{B}_{-\infty}$ and (ii) there are Borel sets A, B and a nonnegative measure ϕ such that $P(B), \phi(A) > 0$ and for all $x \in B$ and Borel $A' \subset A$ one has $P(x, A') \geq \phi(A')$. It is obvious that if $\{x_n\}$ has the representation (2) $\mathcal{B}_{-\infty}$ is trivial.

Bernoulli processes, K-automorphisms, and the T, T^{-1} transformation

Our object is to provide a more detailed discussion than that given in Rosenblatt (2009) showing that the T, T^{-1} transformation provides a counterexample to Wiener's conjecture.

Let us first note that the condition (2) for a stationary process $\{X_n\}$ is natural to consider as a nonlinear version of having the process purely nondeterministic.

An invertible measure preserving transformation T acting on a probability measure space $(M, \mathcal{M}, \mu)$ (with $\mathcal{M}$ a σ-algebra of subsets of M), $\mu(TA) = \mu(T^{-1}A) = \mu(A)$ for $A \in \mathcal{M}$ is called an automorphism. Here $(M, \mathcal{M}, \mu)$ is assumed to be a Lebesgue space. Let X be the set of real numbers, $\mathcal{B}$ the collection of Borel subsets of X and λ a probability measure on $\mathcal{B}$. Take M to be the set of all sequences $x = \{x_n\}$ with $x_n \in X$. Let T be the shift on M, $Tx = x'$, with $x'_n = x_{n+1}$. And set μ on M equal to the product measure of λ on $\mathcal{M}$. The shift T is called a Bernoulli automorphism acting on the Bernoulli sequence.

The case discussed most often is that in which λ has support on a countable set of points of X with the measure λ having finite entropy. Ornstein refers to the case in which the entropy is infinite as a generalized Bernoulli shift.

Consider automorphisms T_1 and T_2 acting on $(M_1, \mathcal{M}_1, \mu_1)$ and $(M_2, \mathcal{M}_2, \mu_2)$ respectively with μ_1 and μ_2 invariant measures on the spaces. The two systems are said to be isomorphic if there is an isomorphism $\phi : (M_1, \mathcal{M}_1, \mu_1) \to (M_2, \mathcal{M}_2, \mu_2), \phi^{-1} : (M_2, \mathcal{M}_2, \mu_2) \to (M_1, \mathcal{M}_1, \mu_1)$ such that $T_2 \phi x^{(1)} =$

$\phi T_1 x^{(1)}$, $T_1\phi^{-1}x^{(2)} = \phi^{-1}T_2 x^{(2)}$ for all $x^{(1)} \in M_1, x^{(2)} \in M_2$. It is understood that these relations are to hold almost everywhere. Any system isomorphic to a Bernoulli system is also called a Bernoulli system.

The entropy of a finite partition P of the space with elements P_i is

$$H(P) = -\sum \mu(P_i) \log \mu(P_i)$$

The entropy per unit time of the partition P with respect to the automorphism T is

$$h(T,P) = \lim_{n\to\infty} \frac{1}{n} H(P \vee T^{-1}P \vee \cdots \vee T^{-n+1}P) \tag{3}$$

($\bigvee_{k=0}^{n-1} T^{-k}P$ is the partition generated by the intersections of $T^{-i}P, i = 0, \ldots, n-1$.)

The entropy of the system $\{T^n\}$ is

$$H(T) = \sup h(T,P)$$

with the supremum taken over all finite partitions P of M. This agrees with the entropy as defined earlier for a Bernoulli system in terms of the product measure of λ. The great achievement of the results of Kolmogorov, Sinai, and Ornstein is that Bernoulli automorphisms with the same entropy are shown to be isomorphic (see Ornstein (1974)).

A family of automorphisms called K-automorphisms were once thought to be Bernoulli automorphisms. All Bernoulli automorphisms are K-automorphisms. Ornstein constructed K-automorphisms that are not Bernoulli and these are the first counterexamples. There are several equivalent formulations of K-automorphisms and we shall give one. The automorphism T is called a K-automorphism if there is a σ-subalgebra $\mathcal{C} \subset \mathcal{M}$

such that

$$
\begin{array}{ll}
(1) & T\mathcal{C} \supset \mathcal{C} \\
(2) & \bigvee_{n=-\infty}^{\infty} T^n \mathcal{C} = \mathcal{M} \\
(3) & \bigwedge_{n=-\infty}^{\infty} T^n \mathcal{C} = \mathcal{N}
\end{array}
\tag{4}
$$

with $\mathcal{N}$ the trivial σ-algebra consisting of sets of measures 0 and 1.

Meilijson (1974) showed that a number of transformations which are special kinds of skew-products are K-automorphisms. The "T, T^{-1}" transformation is an example of this kind of skew-product. Set $Q = (1, -1)$ and let the random variables $\{w_i\}_{i \in \mathbb{Z}}$ be independent and identically distributed (iid) with

$$
w_i = \begin{cases} 1 \text{ with probability } \frac{1}{2} \\ -1 \text{ with probability } \frac{1}{2}. \end{cases} \tag{5}
$$

T is the shift $(Tw)_i = w_{i+1}$ for each $w = \{w_i\}_{i \in \mathbb{Z}}$ in $\Omega = Q^z$. Let the transformation S on $\Omega_1 \times \Omega_2$ be set up so that

$$
S((_1w,\ _2w)) = \begin{cases} (T(_1w), T(_2w)) & \text{if } _2w_0 = 1 \\ (T^{-1}(_1w), T(_2w)) & \text{if } _2w_0 = -1 \end{cases} \tag{6}
$$

Further let

$$
(_1w',_2 w')_n = (S^n(_1w,_2 w))_0 \tag{7}
$$

Introduce

$$
X(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{j=0}^{i-1} w_j & \text{if } i > 0 \\ -\sum_{j=-1}^{i} w_j & \text{if } i < 0 \end{cases} \tag{8}
$$

Then one can show that

$$ {}_2w_i' = {}_2w_i,\ {}_1w_i' = {}_1w_{X(i, {}_2w)} \tag{9} $$

The "T, T^{-1}" transformation S is an automorphism. Let C be the σ-algebra generated by the random variables $({}_1w', {}_2w')_n, n \leq 0$. Clearly $S\mathcal{C} \supset \mathcal{C}$ while $S\mathcal{C} \neq \mathcal{C}$. Also $\vee_{n=-\infty}^{\infty} S^n\mathcal{C}$ is equal to the σ-algebra $\mathcal{M}$ determined by $({}_1w, {}_2w)_n, -\infty < n < \infty$. Meilijson shows that for each bounded real $\mathcal{M}$ measurable function f

$$ \sup_g |E(gS^n f) - E(g)E(f)| \to 0 \text{ as } n \to \infty $$

with the sup taken over all real functions g bounded by one in absolute value and measurable with respect to $S^{-1}\mathcal{C}$. By taking the g's indicator functions of sets in $S^{-1}\mathcal{C}$ and the fact that S is an automorphism one can see that $\bigwedge_{n=-\infty}^{\infty} S^n\mathcal{C} = \mathcal{N}$. Thus S is a K-automorphism and the sequence $({}_1w', {}_2w')_n$ is a stationary sequence that is purely nondeterministic.

Meilijson deals with the more general case of ${}_1T$ an automorphism on $({}_1\Omega, {}_1\mathcal{B}, {}_1P), {}_2T$ with η a countable partition of ${}_2\Omega$ whose ${}_2T$ iterates are ${}_2P$ independent and span ${}_2\mathcal{B}$. Given X a one-to-one integer valued function on η S is defined on $({}_1\Omega \times {}_2\Omega, {}_1\mathcal{B} \times {}_2\mathcal{B}, {}_1P \times {}_2P)$ by

$$ S({}_1w, {}_2w) = ({}_1T^{X({}_2w)}\, {}_1w,\ {}_2T\, {}_2w) \tag{10} $$

He shows that if ${}_1T$ is totally ergodic, S is a K-automorphism. Total ergodicity of a measure preserving transformation is ergodicity of all its positive integer powers.

Notice that in looking at T, T^{-1} transformation acting on $({}_1w,\ {}_2w)$

in $\Omega_1 \times \Omega_2$ the second coordinate dictates which way to shift the first coordinate. For this reason the first coordinate is often referred to as the "scenery" and the second coordinate as the "path". So we have a simple example of a "random walk in a random environment".

Factors and B-automorphisms

Let T, T' be automorphisms of the measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ respectively. If there is a homomorphism $\phi : M \to M'$ such that $\phi(Tx) = T'\phi(x)$ for all $x \in M, T'$ is referred to as a factor automorphism of T. Let M be the set of all real sequences $\xi = \{\xi_n\}, \mathcal{M}$ the product σ-algebra of the components ξ_n and μ the product measure generated by the marginal measures of the ξ_n's. Let T be the shift operator acting on the ξ's. Now suppose a stationary process $\{x_n\}$ has the one-sided representation (2) in terms of the ξ vector of iid random variables. Then

$$x_n = f(T^n \xi) \tag{11}$$

and $x = (x_n, n = \dots, -1, 0, 1, \dots)$. Let T_1 be the shift operator on x sequences.M_1 the space of x sequences, $\mathcal{M}_1$ the σ-algebra on x sequences with μ_1 the measure on $\mathcal{M}_1$ induced by $(M, \mathcal{M}, \mu)$. Set

$$\phi(\xi) = (x_n(\xi), n = \dots, -1, 0, 1, \dots).$$

Then

$$\begin{aligned} \phi(T\xi) &= (x_{n+1}(\xi), n = \dots, -1, 0, 1, \dots) \\ &= T_1 \phi(\xi) \end{aligned} \tag{12}$$

and so $\phi : M \to M_1$ is a homomorphism and T_1 a factor automorphism of the Bernoulli automorphism T. But it is known that a factor-automorphism of a Bernoulli automorphism is a Bernoulli automorphism (see Ornstein (1974)).

Kalikow (1982) has shown that the "T, T^{-1}" transformation is not loosely Bernoulli (a weaker condition than Bernoulli) and so not Bernoulli. So "T, T^{-1}" is a K-automorphism but not Bernoulli. We already know that the stationary sequence $({}_1w',\ {}_2w')_n$ is purely nondeterministic by Meilijson's argument. If it had a representation of the form (2), the process would have to be Bernoulli by the argument given on factors of Bernoulli automorphisms. But it can't be Bernoulli by Kalikow's result. Thus we have a purely nondeterministic stationary sequence which doesn't have a representation of the form (2) providing a counterexample to Wiener's conjecture.

The natural question is that of characterizing the class of nondeterministic processes that have a representation of the form (2) and those purely nondeterministic processes that don't have such a representation. Remarks made by Kalikow suggest that if the path is still a Bernoulli symmetric two shift with equal probabilities to left or right and the scenery Bernoulli of arbitrary entropy the corresponding transformation is still a K-automorphism but not Bernoulli.

Some remarks on recent results for random walks in a random environment can be found in den Hollander and Steif (2006).

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