A directed polymer approach to the once-oriented last passage site percolation time constant in high dimensions

Gregory J. Morrow

Aug. 23, 2009.

Abstract

Let η be a real random variable whose logarithmic moment generating function $\lambda(\beta) := \ln(\mathbf{E}\exp(\beta\eta))$ exists for all $\beta > 0$, and also such that $\mathbf{E}|\eta| < \infty$. Let ν_d denote the point to line last passage time constant of a once-oriented first passage site percolation in $d+1$ dimensions. Here once-oriented refers to the condition that it is always the first coordinate of a path that is increased among oriented paths along sites in $\mathbb{Z}^{(d+1)}$. One can relate ν_d to the free energy of a directed polymer model in the random field of i.i.d. copies of η at low temperature (inverse temperature β near infinity). Here we define the partition function of the directed polymer in the random environment $\{\eta(k,y)\}, k \geq 0, y \in \mathbb{Z}^d$, by $Z_n(\beta) = \mathbb{E}\exp(\beta\sum_{k=1}^n \eta(k, S_k))$ for a random walk $\{S_k\}$ in $\mathbb{Z}^d$, where $\mathbb{E}$ denotes the expectation relative to this random walk. The free energy is then given by $f(\beta) := \lim_{n\to\infty}(\ln Z_n(\beta))/n$. The connection between ν_d and $f(\beta)$, namely $\lim_{\beta\to\infty} f(\beta)/\beta = \nu_d$, was already recognized by Comets and Yoshida [6]. Here we emphasize the fact that for a class of distributions with upper tail $\mathbf{P}(\eta > x) = \exp(-xv(x) + \delta(x))$, such that: $v(x) \nearrow \infty$ and $\delta(x) = O(x)$ as $x \to \infty$, $u(x) := xv(x)$ is strictly convex for large x, $\liminf_{x\to\infty} xv'(x) > 0$, and v satisfies a regularity condition that makes $\exp(-u(x))$ convex for large x, we may obtain an asymptotic evaluation of ν_d as $d \to \infty$. These conditions admit the Poisson case where $v(x) = \ln(x) - 1$ and $\delta(x) = O(\ln(x))$. The proof involves a simple linear estimate on the free energy of the directed polymer model, namely, $f(\beta) \geq \nu_d\beta - \ln(2d)$, that is valid for all $d \geq 1$. Our condition on the upper tail of η is reminiscent but more detailed than a similar condition given by Ben-Ari [1]. We show that $\nu_d \sim \mathbf{E}\max(\eta_1, \ldots, \eta_{2d}) \sim U(\ln(2d))$, where U is the inverse of u. In the Poisson and Gaussian cases we obtain $\nu_d \sim \ln(2d)/\ln\ln(2d)$ and $\nu_d \sim \sqrt{2\ln(2d)}$, respectively, as $d \to \infty$. It follows in particular that if a given distribution of the above class is mixed with a distribution with a lighter upper tail in an appropriate sense then the last passage time constant is ruled asymptotically as $d \to \infty$ by the distribution with the heavier upper tail.

1 Introduction

Let η be a real random variable whose logarithmic moment generating function $\lambda(\beta) := \ln(\mathbf{E}\exp(\beta\eta))$ exists for all $\beta > 0$. We assume also that $\mathbf{E}|\eta| < \infty$, and, since adding a constant to η will only affect our time constant by the same additive constant, we also assume that $\mathbf{E}\eta \geq 0$. Let ν_d denote the point to line last passage time constant (see (1.2)) of a once-oriented first passage site percolation in $d+1$ dimensions. Here once-oriented refers to the condition that it is always the first coordinate of a path that is increased among oriented paths along sites in $\mathbb{Z}^{(d+1)}$. One can relate ν_d to the free energy of a directed polymer model in the random field of i.i.d. copies of η at low temperature (inverse temperature β near infinity). Here we define the partition function of the directed polymer in the random environment $\{\eta(k,y)\}, k \geq 0, y \in \mathbb{Z}^d$ by $Z_n(\beta) = \mathbb{E}\exp(\beta \sum_{k=1}^n \eta(k,S_k))$ for a random walk $\{S_k\}$ in $\mathbb{Z}^d$, where $\mathbb{E}$ denotes the expectation relative to this random walk. The free energy is then given by $f(\beta) := \lim_{n\to\infty}(1/n)\ln Z_n(\beta)$. The connection between ν_d and $f(\beta)$, namely $\lim_{\beta\to\infty} f(\beta)/\beta) = \nu_d$, was already recognized by Comets and Yoshida, [6], Sect. 7. Here we emphasize the fact that we may obtain an asymptotic evaluation of ν_d as $d \to \infty$ by using a simple linear estimate on the free energy of the directed polymer model, namely, $f(\beta) \geq \nu_d - \ln(2d)$ (Proposition 1, below), that is valid for all $d \geq 1$. This bound is closely related in spirit to Talagrand's [19] Proposition 1.1.3 (see also [3], Proposition 1.4). To establish our asymptotic evaluation of ν_d for large d we assume a condition on the upper tail of η reminiscent of a similar condition in [1] that turns our estimation into an interesting calculus problem. Indeed we show perhaps not surprisingly that $\nu_d \sim \mathbf{E}\max(\eta_1,\ldots,\eta_{2d})$, which in the Gaussian case may be shown directly to be given asymptotically by $\sqrt{2\ln(2d)}$, as $d\to\infty$. By this method it follows in particular that if a distribution of a certain class is mixed with a distribution with a lighter upper tail then the last passage time constant is ruled asymptotically by the distribution with the heavier upper tail; see the remarks after the proof of Theorem 1 at the end of the paper. Although our method does give some estimates on first passage time constants (see [15]) these turn out to be weaker in the Bernoulli case than can be found by application of [12].

We first introduce the once-oriented first passage site percolation model. Define the cone of sites $\mathbb{K} = \mathbb{K}_{d+1} := \{z = (t,y) \in \mathbb{Z}_+ \times \mathbb{Z}^d : t \geq 0 \text{ and } |y|_1 \leq t\}$, for $y = (y_1,\ldots,y_d) \in \mathbb{Z}^d$ and $|y|_1 := |y_1| + \cdots + |y_d|$. For a non-negative integer n let $\gamma = (\gamma_0,\gamma_1,\ldots,\gamma_n)$ be a sequence of nearest neighbor positions in $\mathbb{Z}^d$, with $\gamma_0 = \mathbf{0} \in \mathbb{Z}^d$. This means that for each $t = 0,1,\ldots,n-1$, the increment $\gamma_{t+1} - \gamma_t = \pm e_i$ for some i^{th} unit coordinate vector e_i in $\mathbb{Z}^d$. The sequence of sites (t,γ_t), $t = 0,1,\ldots,n$, describes a once-oriented path $\overrightarrow{\gamma_n} = \overrightarrow{\gamma_n}(\gamma)$ of length n in the cone $\mathbb{K}$. When $d = 1$ this model is equivalent to the usual directed site percolation on $\mathbb{Z}_+^2$. Here the usual directed site percolation model on $\mathbb{Z}_+^{d+1}$ assumes that exactly one coordinate of a path increases by 1 at each step instead of our assumption that it is always the first coordinate of $\overrightarrow{\gamma_n}$ that must increase. Thus for oriented paths in $d+1 = 3$ and higher space dimensions the two models are not equivalent because in the once-oriented case there are $2d$ choices for each step of the path compared to $d+1$ choices in the usual directed case. Our motivation for the once-oriented case is in fact its relation to the usual directed polymer model that we introduce below. Introduce an i.i.d. field of random variables $\{\eta(z)\}$ over $z \in \mathbb{K}$, where the common distribution of the field is given by η and where we denote the probability and expectation relative to this field by $\mathbf{P}$ and $\mathbf{E}$ respectively. Denote the passage time over $\overrightarrow{\gamma_n}$ by

$$T(\overrightarrow{\gamma_n}) := \sum_{t=1}^{n} \eta((t,\gamma_t)),$$

and define the last passage time to pass from the origin at the vertex of $\mathbb{K}$ to $(n,\mathbf{0}) \in \mathbb{K}$ by

$$\widetilde{T}_{0,n} := \max T(\overrightarrow{\gamma_n}),$$

where the maximum is extended over all once-oriented paths of length n with $\gamma_0 = \gamma_n = \mathbf{0} \in \mathbb{Z}^d$.

We define for any $0 \le m < n$ a once-oriented path $(m, \gamma_0), (m+1, \gamma_1), \ldots, (n, \gamma_{n-m})$ in $\mathbb{K}$, with $(\gamma_0, \gamma_1, \ldots)$ as above, and define $\widetilde{T}_{m,n} := \max_{\gamma_0=\gamma_{n-m}=\mathbf{0}} \sum_{t=1}^{n-m} \eta((m+t, \gamma_t))$. Then obviously the collection of times $\{\widetilde{T}_{m,n}\}$ is superadditive. By superadditivity we have that there exists a constant μ_d, called the point to point last passage time constant, such that

$$\lim_{n\to\infty} \mathbf{E}\widetilde{T}_{0,n}/n = \mu_d = \sup_n \mathbf{E}\widetilde{T}_{0,n}/n,$$

and, by Kingman's ergodic theorem, $\lim_{n\to\infty} \widetilde{T}_{0,n}/n = \mu_d$, $\mathbf{P}$-a.s. and in $\mathbf{L}^1$.

We also introduce the "origin to line" last passage times

$$\widetilde{H}_{m,n} := \max_{\gamma_0=\mathbf{0}} \sum_{t=1}^{n-m} \eta((m+t, \gamma_t)), \tag{1.1}$$

where the nearest neighbor sequence of positions $(\gamma_0, \gamma_1, \ldots)$ in $\mathbb{Z}^d$ is as before, but there is no condition on the "height" γ_{n-m}. Even though the collection $\{\widetilde{H}_{m,n}\}$ is not superadditive, so that Kingman's ergodic theorem does not directly apply, the collection of expectations $\{\mathbf{E}\widetilde{H}_{0,n}\}$ is superadditive, so that the corresponding point to line last passage time constant

$$\nu_d = \nu_d(\eta) := \lim_{n\to\infty} \mathbf{E}\widetilde{H}_{0,n}/n = \sup_{n\ge 1} \mathbf{E}\widetilde{H}_{0,n}/n \tag{1.2}$$

is well defined. We may want to know whether the point to line time constant ν_d is equal to the point to point time constant μ_d. One way to determine this is to use the shape theorem of Martin [13], Thm. 5.1, since as pointed out in [11] we have that the shape theorem implies $\nu_d = \mu_d$. Note that Martin studies the usual directed percolation on $\mathbb{Z}_+^{d+1}$; the proofs for the once-oriented case are obtained in the same way by the concentration inequality Lemma 3.1 of [13]. Yet, due to its simplicity, it is instructive to see the equality of μ_d and ν_d for the present oriented case by the original approach of Hammersley and Welsh [8] as shown in Smythe and Wierman [18], Thm. 5.3. For completeness we summarize briefly this argument as follows. Since by definition we already have $\mu_d \le \nu_d$, the basic idea is to show that for each $k \ge 1$,

$$\mathbf{E}\widetilde{H}_{0,k}/k \le \mu_d. \tag{1.3}$$

To do this, for a given configuration we take successive point to line last-passage-time oriented routes over intervals in the oriented x-direction of length k, so that successive routes span successive congruent small cones connected vertex to base along the oriented direction via successive translations of the initial small cone $K^{(0)} \subset \mathbb{K}$, where $K^{(0)} := \{(x, y) \in \mathbb{K}_{d+1} : 0 \le x \le k, |y|_1 \le x\}$. The vertex of the second small cone is placed at the position $(k, \mathbf{h}^{(0)})$, where $\mathbf{h}^{(0)}$ is the "height" such that the initial point to line last passage time $\widetilde{H}_{0,k}^{(0)} := \widetilde{H}_{0,k}$ over $K^{(0)}$ is achieved for $(k, \mathbf{h}^{(0)})$ as the final point on a path with, say, smallest 1-norm $|\mathbf{h}^{(0)}|_1$. Note that $\mathbf{h}^{(0)}$ has a symmetric distribution in $\mathbb{Z}^d$. Coming back now to the original configuration, we denote the point to line last passage time across the second small cone by $\widetilde{H}_{k,2k}^{(1)} := \max_{\gamma_0=(k,\mathbf{h}^{(0)})} \sum_{t=1}^{k} \eta((k+t, \gamma_t))$, and we denote by $(2k, \mathbf{h}^{(0)} + \mathbf{h}^{(1)})$ the final point along a path of length k for which $\widetilde{H}_{k,2k}^{(1)}$ is achieved, again say with smallest 1-norm $|\mathbf{h}^{(1)}|_1$. Continuing in this way we construct n such successive passage times

$\widetilde{H}^{(j)}_{jk,jk+k}$, $j = 0, 1, \ldots, n-1$, and corresponding i.i.d. heights $\mathbf{h}^{(j)}$. In the present model the heights are bounded in Euclidean norm, $\|\mathbf{h}^{(j)}\| \leq \sqrt{d}k$, so that in particular $\sqrt{\mathbf{E}\|h^{(0)}\|^2} \leq \sqrt{d}k$. Therefore, since the mean square of the Euclidean norm of the sum of these heights is given by $n\mathbf{E}\|\mathbf{h}^{(0)}\|^2$, we have that the 1-norm of the average height $L_n := (\mathbf{h}^{(0)} + \mathbf{h}^{(1)} + \cdots + \mathbf{h}^{(n-1)})/n$ converges to zero in mean: $\mathbf{E}|L_n|_1 \to 0$ as $n \to \infty$. Finally construct an oriented path $\overrightarrow{r_n}$ from (kn, nL_n) back to $(kn + n|L_n|_1, \mathbf{0}) \in \mathbb{K}$ of length $n|L_n|_1$, to obtain an estimate

$$\sum_{j=0}^{n-1} \widetilde{H}^{(j)}_{jk,jk+k} + T(\overrightarrow{r_n}) \leq \widetilde{T}_{0,kn+n|L_n|_1} \tag{1.4}$$

Now divide (1.4) by kn. Since $\widetilde{T}_{0,kn+n|L_n|_1}/(kn) \leq (\sup_N \widetilde{T}_{0,N}/N)(1 + |L_n|_1/k) = \mu_d(1 + |L_n|_1/k)$, $\mathbf{P}$-a.s., and since $\mathbf{E}T(\overrightarrow{r_n})/(kn) = (\mathbf{E}\eta)\mathbf{E}|L_n|_1/k \to 0$ as $n \to \infty$, then (1.3) follows after taking the expectation $\mathbf{E}$ on both sides and letting $n \to \infty$. Thus after taking $k \to \infty$ in (1.3) we find also $\nu_d \leq \mu_d$.□

In the special cases of the exponential and geometric distributions, the last passage time constants have been evaluated explicitly for $d + 1 = 2$ as follows. If η is unit exponential then $\nu_1(\eta) = 2$. If η is geometric (p), then $\nu_1(\eta) = (1 + \sqrt{1-p})/p$. For both these special cases a Shape Theorem holds:

$$\lim_{n\to\infty} (1/n)\widetilde{T}_{0,(nx,ny)} = g(x, y)$$

where, formulated on $\mathbb{Z}^2_+$, if $\eta \sim$ exp, then $g(x, y) = (\sqrt{x} + \sqrt{y})^2$, and if $\eta \sim$ geo(p), then $g(x, y) = \left(x + y + 2\sqrt{xy(1-p)}\right)/p$; see [14]. By setting $x = y = 1/2$ we obtain the values for the time constants ν_1 above. Also in these special cases $(\widetilde{T}_{0,n} - n\nu_1)/n^{1/3}$ converges in distribution to the Tracy-Widom law F_2, [10]. Thus in these cases and for $d + 1 = 2$ (the 2 dimensional case), the variance of the last passage times $\widetilde{T}_{0,n}$ grows only at the rate $n^{2/3}$. It is unknown how this variance grows in higher dimensions. For related results in the 2 dimensional case see [16].

Our method hinges on the relationship of last passage times to the limiting behavior in the limit of low temperature of the directed polymer model. We now introduce this polymer model; see [5] for a physical motivation. We define the (random) partition function Z_n at level n as a function of the inverse temperature $\beta > 0$ by

$$Z_n(\beta) := (2d)^{-n} \sum_{\gamma} \exp(\beta T(\overrightarrow{\gamma_n})),$$

where the sum is over all $(2d)^n$ nearest neighbor sequences $(\gamma_0, \gamma_1, \ldots, \gamma_n)$ in $\mathbb{Z}^d$ starting from the origin as above. An alternative representation of the partition function is given by

$$Z_n(\beta) = \mathbb{E} \exp(\beta \sum_{k=1}^{n} \eta(k, S_k)),$$

where $\{S_k\}$ is a simple symmetric random walk in $\mathbb{Z}^d$ and where $\mathbb{E}$ denotes the expectation with respect to this random walk. Thus $Z_n(\beta)$ is the partition function of a directed polymer in a random environment; the environment is the field $\{\eta(z)\}$ and, in the formula $Z_n = \text{ave.}\exp[-\text{energy}]$, the energy of the path $\overrightarrow{\gamma_n}$ is simply: energy $= -\beta T(\overrightarrow{\gamma_n})$. Finally we define the (random) Gibbs measure on the collection of nearest neighbor sequences $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ by

$$\mu_n(\gamma) := (2d)^{-n} \exp(\beta T(\overrightarrow{\gamma_n}))/Z_n(\beta),$$

and write $\widetilde{\mathbb{E}}$ for the integration operation with respect to μ_n. When $\beta = 0$ we simply obtain uniform measure on γ (independent of the environment) that is of course the measure of the simple symmetric random walk. In a series of papers going back to [2], where Bolthausen gives a martingale proof of the original result of Imbrie and Spencer [9] on the diffusive behavior of the height γ_n for small $\beta > 0$ and high dimensions ($d+1 \geq 4$), namely $\widetilde{\mathbb{E}}|\gamma_n|^2/n \to 1$, $\mathbf{P}$-a.s., a martingale theory has been established for the directed polymer in a random environment (see [5],[6]). Here we emphasize that the logarithmic moment generating function $\lambda(\beta) := \ln \mathbf{E} \exp(\beta\eta)$ is assumed to exist for all $\beta \geq 0$. Obviously we have that $\mathbf{E} Z_n(\beta) = \exp(n\lambda(\beta))$. It turns out that the martingale theory leads to a dichotomy: for all $\beta \geq 0$, $\lim_{n\to\infty} Z_n(\beta)\exp(-n\lambda(\beta)) = \widetilde{Z}_\infty(\beta)$ exists, and either

$$\text{(i) } \widetilde{Z}_\infty(\beta) > 0 \ \ \mathbf{P}\text{-a.s. or, (ii) } \widetilde{Z}_\infty(\beta) = 0 \ \ \mathbf{P}\text{-a.s.}$$

In case (i) it is said that weak disorder holds, while under (ii) it is said that strong disorder holds. For example weak disorder holds if $d+1 \geq 4$ and $\beta > 0$ is small, and, further, if (ii) holds for some $\beta > 0$ then there is a critical inverse temperature β_c above which the strong disorder holds [6]. For $d = 1$, Comets and Yoshida [6] show that by contrast with the high dimensional case, $\beta_c = 0$, so that in the plane there is sufficient interaction among the oriented paths to yield the strong disorder no matter how high the temperature.

By writing $\ln Z_{n+m} = \ln Z_n + \ln \sum_y \mu_n(\gamma_n = y) Z^y_{n,m}$, one finds by Jensen's inequality and the fact that $\mathbf{E} \ln Z^y_{n,m} = \mathbf{E} \ln Z_m$, that $\mathbf{E} \ln Z_n$ is superadditive (see [4], Proposition 1.5). Hence by superadditivity the so-called free energy $f(\beta)$ exists as follows:

$$\lim_n \mathbf{E}(\ln Z_n)/n = f(\beta) = \sup_n \mathbf{E}(\ln Z_n)/n. \tag{1.5}$$

Jensen's inequality applied with respect to $\mathbf{E}$ immediately gives that $f(\beta) \leq \lambda(\beta)$. By [4] one also obtains a concentration inequality: $\mathbf{P}(|(1/n)\ln Z_n - f(\beta)| > \delta) \leq \exp(-\delta^2 n/c)$, so that in particular by the Borel-Cantelli lemma,

$$\lim_{n\to\infty} (1/n) \ln Z_n = f(\beta) \ \ \mathbf{P}\text{-a.s.} \tag{1.6}$$

It is well known that $f(\beta)$ is convex in β (see below). Further it is shown by [6] using an FKG inequality that the so-called Lyapunov exponent $\psi(\beta) := \lambda(\beta) - f(\beta)$ is non-decreasing. Comets and Yoshida ([6], Theorem 3.2) use this monotonicity property together with the martingale theory developed in [5] to establish that a phase transition from weak disorder to strong disorder occurs at most once and that $\psi(\beta) = 0$ for $0 \leq \beta \leq \beta_c$. For $d \geq 2$ an open problem is whether there exists a second phase transition $\beta_c^\psi > \beta_c$ such that $\psi(\beta) = 0$ for $\beta < \beta_c^\psi$ and $\psi(\beta) > 0$ for $\beta > \beta_c^\psi$ (see [6], Remark 3.2). In $d = 1$ Comets and Vargas [7] have shown that there is no such second transition so that $\psi(\beta) > 0$ for all $\beta > 0$.

2 Bound on the free energy

Our object in this Section is to obtain a bound on the free energy (Proposition 1, below) that we can then apply to the problem of estimating ν_d in general. To see how the free energy is related to the time constant, consider in essence the double limit $\lim_{n,\beta\to\infty}(\ln Z_n(\beta))/(n\beta)$ as in [6], Sect. 7. We proceed as follows. First estimate $\sum_\gamma \exp(\beta T(\overrightarrow{\gamma_n}))$ from below by the maximum term:

$$\mathbf{E}(\ln Z_n) = \mathbf{E}\ln[(2d)^{-n}\sum_\gamma \exp(\beta T(\overrightarrow{\gamma_n}))] \geq \beta\mathbf{E}\widetilde{H}_{0,n} - n\ln(2d).$$

Next estimate also the average above by the largest term:

$$\beta\mathbf{E}\widetilde{H}_{0,n} = \max_\gamma(\beta T(\overrightarrow{\gamma_n})) \geq \mathbf{E}(\ln Z_n).$$

By dividing by n and taking $n\to\infty$, in each of these last two relations, and by recalling the definitions (1.2) and (1.5), we have thus proved the following.

Proposition 1 *We have the following linear lower bound for the free energy:*

$$f(\beta) \geq \beta\nu_d - \ln(2d), \text{ for all } \beta \geq 0.$$

Furthermore,

$$\lim_{\beta\to\infty} f(\beta)/\beta = \nu_d = \sup_{n\geq 1}\mathbf{E}\widetilde{H}_{0,n}/n.$$

We remark that a slightly different approach to Proposition 1 similar in spirit to that above may be made by working with quantities before taking expectations under the assumption that the exponential moment $\mathbf{E}\exp(\beta\eta)$ exists for all $\beta>0$. Indeed, since there is always at least one oriented path along which $\widetilde{H}_{0,n}$ is attained, we have the simple estimate

$$\ln[\exp(\beta\widetilde{H}_{0,n})1/(2d)^n]/(n\beta) \leq \ln Z_n(\beta)/(n\beta) \leq \widetilde{H}_{0,n}/n.$$

Therefore, we have for all $n\geq 1$ and $\beta>0$

$$\widetilde{H}_{0,n}/n - \ln(2d)/\beta \leq \ln Z_n(\beta)/(n\beta) \leq \widetilde{H}_{0,n}/n. \tag{2.1}$$

Thus the conclusion of Proposition 1 follows from (1.6) and (2.1) once we can prove that

$$\lim_{n\to\infty}\widetilde{H}_{0,n}/n = \nu_d \ \mathbf{P}\text{-a.s.} \tag{2.2}$$

But since under the exponential moment assumption we have in particular that the second moment of $\widetilde{H}_{0,n}$ exists for all n, we have that (2.2) follows from Cor. 5.5 and Thm. 2.9 of [18].

In the Gaussian case ($\eta\sim\mathcal{N}(0,1)$) we have by [3] that for $\beta>0$,

$$f(\beta) \leq \min\left(\beta^2/2, \beta\sqrt{2\ln(2d)}\right), \tag{2.3}$$

so in this case one immediately obtains by Proposition 1 that

$$\nu_d(\mathcal{N}(0,1)) \leq \sqrt{2\ln(2d)}. \tag{2.4}$$

Remark 1 *One can show directly (see [15])that* $\mathbf{E}\max(g_1,\ldots,g_{2d}) = \sqrt{2\ln(2d)} + o(1)$ *as* $d\to\infty$*, where* $g_1,\ldots,g_{2d}$ *denote i.i.d.* $\mathcal{N}(0,1)$ *r.v.'s. So by (1.2) with* $n=1$ *and (2.3) we obtain in the Gaussian case that likewise,* $\nu_d = \sqrt{2\ln(2d)} + o(1)$ *as* $d\to\infty$.

In Theorem 1, below, we extend this Gaussian example to an asymptotic evaluation of ν_d for a certain class of distributions with upper tails smaller than the tail of the exponential distribution.

Finally we note in this vein that even though we asymptotically evaluate ν_d in Theorem 1 below for a class of distributions such that $\nu_d \to \infty$ as $d \to \infty$, it is still open to determine whether in any given case $\nu_d > 0$ for small d. Whenever this is true then, since $\mu_d = \nu_d$, we also get the shape theorem as in [13].

It is well known that $f_n(\beta) := \mathbf{E} \ln Z_n(\beta)$ is convex (since the second derivative in β is a variance with respect to Gibbs measure). Since $\mathbf{E}\eta \geq 0$, we have that $f_n(\beta)$ is non-negative and non-decreasing for $\beta \geq 0$. Indeed for $\Delta\beta := \beta_1 - \beta_0 > 0$

$$f_n(\beta_1) = f_n(\beta_0) + (1/n)\mathbf{E} \ln \mathbb{E} \exp(\Delta\beta T) \left[\exp(\beta_0 T)/\mathbb{E} \exp(\beta_0 T)\right]$$

$$= f_n(\beta_0) + (1/n)\mathbf{E} \ln \widetilde{\mathbb{E}} \exp(\Delta\beta T) \geq f_n(\beta_0) + \mathbf{E}\widetilde{\mathbb{E}}(\Delta\beta)T/n \geq f_n(\beta_0),$$

where at the last step Jensen's inequality was applied w.r.t. $\widetilde{\mathbb{E}}$. Here $\mathbb{E}$ is the uniform measure on directed paths and $\widetilde{\mathbb{E}}$ is simply the Gibbs measure expectation. Also we have written T for short in place of $T(\overrightarrow{\gamma}_n)$. Thus taking limits in (1.5) we obtain these properties for $f(\beta)$. Thus the Proposition 1 simply gives a bound on the constant C_d for the tangent line $\lambda = \nu_d\beta + C_d$ of $f(\beta)$ at $\beta = \infty$; namely, $C_d \geq -\ln(2d)$. Notice that in the Gaussian case we have that $\beta_c^\psi \leq \nu_d$ by $\lambda'(\beta) = \beta$ and convexity of $f(\beta)$.

We also remark that the inequality (2.4) may be obtained from the convexity of $f(\beta)$ and $f(\beta) \leq \lambda(\beta) = \beta^2/2$ using Proposition 1 as follows. The tangent line to the graph of $\lambda(\beta)$ at $\beta = \nu_d$ is given by $\lambda = \nu_d\beta - \nu_d^2/2$, with slope ν_d. Since the graph of $\lambda(\beta)$ lies above that of $f(\beta)$ which in turn by Proposition 1 lies above the line $\lambda = \nu_d\beta - \ln(2d)$, we have that the λ-intercepts of the two lines are in the relation $-\nu_d^2/2 \geq -\ln(2d)$, so (2.4) follows. In addition, by Remark 1 the constant $\ln(2d)$ in Proposition 1 is asymptotically best possible as $d \to \infty$ in the Gaussian case.

We conclude this section by illustrating Proposition 1 to obtain an asymptotic evaluation of the last passage time constant in the unit exponential case. The novelty of our approach is its simplicity. First define

$$m_d := \mathbf{E} \max(\eta_1, \ldots, \eta_{2d}). \tag{2.5}$$

Since $m_d = \mathbf{E}\widetilde{H}_{0,1}$, we have by (1.2) that $\nu_d \geq m_d$. We show that in fact $\nu_d \sim m_d$ as $d \to \infty$, as follows. First write $G(t) := t^{2d}$ for $t = t(x) := \mathbf{P}(\eta \leq x) = 1 - \exp(-x)$, $x \geq 0$. By the change of variables $t = t(x)$ we have $m_d = \int_0^1 -\ln(1-t)\mathrm{d}G(t)$. But $-\ln$ is a convex function. So by Jensen's inequality we find $m_d \geq -\ln(\int_0^1 (1-t)\mathrm{d}G(t)) = -\ln(1 - 2d/(2d+1)) = \ln(2d) + o(1)$ as $d \to \infty$. Next we use Proposition 1 to show also that $\nu_d \leq \ln(2d)(1 + o(1))$. We have that

$$\lambda(\beta) \geq f(\beta) \geq \beta\nu_d - \ln(2d). \tag{2.6}$$

Now $\lambda(\beta) = -\ln(1-\beta)$, $\beta < 1$. Choose $\beta = 1 - 1/\ln(2d)$. By (2.6) we therefore have that $\ln\ln(2d) \geq \nu_d(1 - 1/\ln(2d)) - \ln(2d)$. It follows that $\nu_d \leq \ln(2d) + O(\ln\ln(2d))$. Hence we find that in the unit exponential case

$$\nu_d = \ln(2d)(1 + o(1)) \text{ as } d \to \infty.$$

3 Asymptotic evaluation of ν_d as $d \to \infty$

In this section we generalize our asymptotic evaluation of the last passage time constant ν_d by assuming a nice form of the upper tail of the distribution of η, namely conditions (3.4)-(3.7) below, that appears in the same guise but in less detailed form in [1]. We start with a series of assumptions on the distribution of η, mainly that the upper tail decays at least exponentially, but that will also include the discrete case, to get a lower bound for the moment m_d of (2.5). Denote the distribution function $F(x)$ of η by $F(x) := \mathbf{P}(\eta \leq x)$. We assume that $F(x)$ takes the form $F(x) = 1 - \exp(-u(x))$, where $u(x)$ is non-decreasing. We also assume that there exist two distribution functions $F_i(x) := 1 - \exp(-u_i(x))$, $i = 0, 1$, such that the associated exponents $u_i(x)$ are strictly increasing continuously differentiable positive functions for all $x \in [x_0, \infty)$, with some $x_0 > 0$, such that for $x \geq x_0$, $u(x)$ falls between these two exponents and is asymptotic to each at infinity:

$$u_0(x) \leq u(x) \leq u_1(x), \text{ with } u_i'(x) > 0, \quad x \geq x_0, \text{ and } u_1(x) = (1 + o(1))u_0(x), \text{ as } x \to \infty. \tag{3.1}$$

Note that the upper tail for exponent u_1 is smaller than the upper tail for exponent u_0, so $F_1(x) \geq F_0(x)$, $x \geq x_0$, and a random variable under F_1 will take smaller values x near infinity than another variable under F_0 under a coupling for these distributions. We also assume the exponential tail condition that

$$\liminf_{x\to\infty} u_i'(x) > 0, \quad i = 0, 1. \tag{3.2}$$

Finally we assume that each of the distribution functions $F_i(x)$ is concave for $x \geq x_0$, a condition that is equivalent under the existence of second derivatives of the u_i to:

$$-u_i'^2(x) + u_i''(x) \leq 0, \quad x \geq x_0. \tag{3.3}$$

Define U as the function inverse of u; $U := u^{-1}$, so that $U(u)$ is nondecreasing to ∞, for $u \geq u(x_0)$.

Lemma 1 *Assume the conditions* (3.1)-(3.3) *and also that* $\mathbf{E}|\eta| < \infty$*. Then we have that*

$$m_d \geq U(\ln(2d))(1 + o(1)), \text{ as } d \to \infty.$$

For convenience, before we prove this lemma we also state here the exponential moment condition that we will use to prove our Theorem 1 below. Assume that the exponent $-u(x)$ of the distribution function $F(x)$ is given by the form

$$u(x) = xv(x) + \delta(x), \; x \geq x_0, \tag{3.4}$$

where $v(x)$ is twice continuously differentiable and satisfies the following shape conditions:

$$v(x), \quad x \geq x_0, \text{ is strictly increasing to } \infty, \text{ and } xv(x), \quad x \geq x_0, \text{ is strictly convex,} \tag{3.5}$$

as well as the following growth condition:

$$\liminf_{x\to\infty} xv'(x) > 0. \tag{3.6}$$

We assume that $\delta(x) = O(x)$, as $x \to \infty$. Finally we assume that v satisfies a regularity condition:

$$\liminf_{x\to\infty} (xv(x))'^2/(xv(x))'' > 1. \tag{3.7}$$

where the prime denotes differentiation with respect to x. After a bit of computation with $u_0(x) := xv(x) - Mx$ and $u_1(x) := xv(x) + Mx$ for a suitably large constant $M > 0$, we can see that (3.1)-(3.2) hold by (3.4) and (3.6). Further (3.3) holds in addition by applying (3.7) in conjunction with (3.4) and (3.6). Thus (3.4), (3.6), and (3.7) are sufficient for Lemma 1. We will use condition (3.5) together with (3.4) and (3.6) to prove Lemma 2 below.

We define V as the function inverse of v; $V := v^{-1}$, so $V(v)$ increases to ∞, for $v \geq v(x_0)$. Notice that the exponential and Gaussian cases correspond to $v(x) = 1$ and $v(x) = x/2$, or $U(u) = u$ and $U(u) = \sqrt{2u}$, respectively, even though the exponential case does not satisfy (3.6). Furthermore the Poisson case corresponds to $v(x) = \ln(x) - 1$ and $\delta(x) = O(\ln(x))$, so that $U(u) \sim u/\ln(u)$ as $u \to \infty$. By the growth condition (3.6) we can see that the Poisson distribution is a boundary case for Theorem 1.

Proof of Lemma 1. The proof is by an application of Jensen's inequality in a similar way as shown in the example at the end of the last paragraph of Section 2. As before denote by $F(x)$ the cumulative distribution function of η: $F(x) := \mathbf{P}(\eta \leq x)$. We have that $m_d = \int_{-\infty}^{\infty} x \mathrm{d}F^{2d}(x)$. Now we make the substitution $t = F(x)$ to write the last integral as $m_d = \int_0^1 F^{-1}(t)\mathrm{d}(t^{2d})$. Denote $t_0 := F(x_0)$ and note that $t_0 < 1$. We split the last integral for m_d as $\left(\int_0^{t_0} + \int_{t_0}^1\right) F^{-1}(t)\mathrm{d}(t^{2d}) = I + II$. Rewrite $I = \int_{-\infty}^{x_0} 2dxF^{2d-1}(x)\mathrm{d}F(x)$, and assume that $d \geq 2$ since we are eventually going to take $d \to \infty$. We note that due to the assumption $\mathbf{E}|\eta| < \infty$, we have that $M_0 := \sup_{x \leq x_0} |xF(x)| < \infty$. Hence we may estimate that $|I| \leq M_0 \int_{-\infty}^{x_0} 2dF^{2d-2}(x)\mathrm{d}F(x) = (2d/(2d-1))M_0 t_0^{2d-1} = o(1)$ as $d \to \infty$. We next estimate II from below by first using the left hand inequality in $F_1^{-1}(t) \leq F^{-1}(t) \leq F_0^{-1}(t)$, $t \geq t_0$, and second by using that $F_1^{-1}(t)$, is convex on $[t_0, 1)$ by (3.3). Hence by writing $II \geq (1 - t_0^{2d}) \int_{t_0}^1 F_1^{-1}(t)\mathrm{d}\left(t^{2d}/(1 - t_0^{2d})\right)$, we have by Jensen's inequality applied to F_1^{-1} that

$$II \geq (1 - t_0^{2d}) F_1^{-1}\left(\int_{t_0}^1 t\mathrm{d}\left(t^{2d}/(1 - t_0^{2d})\right)\right).$$

Hence, putting together our estimates we have shown that

$$m_d \geq o(1) + (1 + o(1))F_1^{-1}\left((2d/(2d+1))(1 + O(t_0^{2d}))\right), \text{ as } d \to \infty \tag{3.8}$$

It remains to get a suitable estimate for $F_1^{-1}(t)$ as $t \to 1$. For $t \geq t_0$ let us write $t = 1 - p$ for $p = p(x) := \exp(-u_1(x))$. Therefore $-\ln(p(x)) = u_1(x)$, and $x = U_1(-\ln(p(x)))$. Now set $p = (1 + O(t_0^{2d}))/(2d+1)$ and find that $x = U_1(\ln(2d) + O(1/d))$. Finally by condition (3.2), we have that $u_1'(x) \geq c > 0$, for all large x and some constant $c > 0$, so that $U_1'(u) \leq (1/c)$, for all large u. Therefore $U_1(\ln(2d) + O(1/d)) = U_1(\ln(2d)) + O(1/d) = (1 + o(1))U_1(\ln(2d))$. Thus also $II \geq (1 + o(1))U_1(\ln(2d))$. Finally, since the term $I = o(1)$ is clearly $o(1)U_1(\ln(2d))$, to complete the proof we need only establish that $U(u) \sim U_1(u)$, as $u \to \infty$. However the growth condition (3.2) again implies that the change in x, $\Delta x > 0$, such that $u_0(x + \Delta x) \geq u_0(x)(1 + \epsilon)$ satisfies $\Delta x \leq (\epsilon/c)x$ for a constant $c > 0$ and all large x. Hence this gives that $U_1(u) = (1 + o(1))U_0(u)$ as $u \to \infty$. Since we also have that $U_0(u) \geq U(u) \geq U_1(u)$ for all large u, this completes the proof. □

The object of the present section is to prove the following.

Theorem 1 *Assume that conditions* (3.4)-(3.7) *hold and that* $\mathbf{E}|\eta| < \infty$. *Then the once-oriented last passage time constant* ν_d *satisfies*

$$\nu_d \sim U(\ln(2d)), \ \ \textit{as } d \to \infty.$$

Remark 2 *By Lemma 1 and the relation* $\nu_d \geq m_d$*, it follows by Theorem 1 that we may also write* $\nu_d \sim m_d$ *as* $d \to \infty$.

Before we can prove this theorem, we will need an estimate of the logarithmic moment generating function $\lambda(\beta)$. To simplify the computations we define a random variable η_+ taking values in $[x_0, \infty)$ to have the distribution function $F_+(x) := \mathbf{P}(\eta \leq x | \eta \geq x_0)$, so that

$$F_+(x) = 1 - C_0 \exp(-u(x)), \quad x \geq x_0, \tag{3.9}$$

for $C_0 = \exp(u(x_0))$. Then since η_+ is stochastically larger than η, we have that $\nu_d(\eta_+) \geq \nu_d(\eta)$. Thus by using $\lambda_+(\beta) \geq f_+(\beta) \geq \nu_d(\eta_+)\beta - \ln(2d) \geq \nu_d(\eta)\beta - \ln(2d)$, where f_+ denotes the free energy of η_+, it will suffice to estimate the logarithmic moment generating function $\lambda_+(\beta)$ of η_+ at an appropriately chosen value of β. Denote:

$$w(x) := (xv(x))' = xv'(x) + v(x), \tag{3.10}$$

and note that $w(x) \nearrow \infty$. Denote also, with a slight abuse of notation, $U(u)$ as the function inverse of the nice exponent $u = xv(x)$.

Lemma 2 *Assume that the conditions* (3.4)-(3.6) *hold. Then for* $\beta = \beta_d := w(U(\ln(2d))) - M$*, for some constant* $M > 0$*, we have that* $\lambda_+(\beta_d) \leq (U^2 v'(U))(\ln(2d))(1 + o(1))$*, as* $d \to \infty$.

Proof of Lemma 2. We write by integration by parts that

$$\exp(\lambda_+(\beta)) = -\int_{x_0}^{\infty} \exp(\beta x) \mathrm{d}(1 - F_+(x)) = -[\exp(\beta x)(1 - F_+(x))]_{x_0}^{\infty} + C_0 \beta \int_{x_0}^{\infty} \exp(\beta x - u(x)) \mathrm{d}x,$$

where the integrated term is zero by (3.5). Hence by substituting for $u(x)$ by (3.4), we have

$$\exp(\lambda_+(\beta)) = C_0 \beta \int_{x_0}^{\infty} \exp(\beta x - xv(x) + \delta(x)) \mathrm{d}x.$$

Substitute $v = v(x)$ or $x = V(v)$, to rewrite this integral as

$$\exp(\lambda_+(\beta)) = C_0 \beta \int_{v(x_0)}^{\infty} \exp(-(v - \beta)V(v) + \delta(V(v)))V'(v) dv.$$

Break this integral into two pieces, $I + II$, where

$$I := C_0 \beta \int_{v(x_0)}^{2\beta} \exp(-(v - \beta)V(v) + \delta(V(v)))V'(v) dv,$$

and $II := \exp(\lambda_+(\beta)) - I$. Now, using that the term $(v - \beta)V(v) \geq V(v)$ for $v \geq 2\beta$, and that $V' > 0$, we immediately obtain, using $-\beta V(v) + \delta(V(v)) \leq -(\beta/2)V(v)$ for large v, that

$$II = C_0 \beta \int_{2\beta}^{\infty} \exp(-V(v) + \delta(V(v)))V'(v) dv \leq 2C_0 \exp(-(\beta/2)V(2\beta)) = o(1), \ \text{as } \beta \to \infty.$$

It remains to estimate I. We have the simple estimate obtained by maximizing the exponent as follows:

$$I \le C_0 \beta \exp\left(\max_{v(x_0)\le v\le 2\beta} \{V(v)(\beta + M - v)\}\right)[V(2\beta) - x_0], \tag{3.11}$$

for a suitably large constant $M > 0$ such that $|\delta(x)| \le Mx$, $x \ge x_0$. The exponent to be maximized is of course none other than $(\beta+M)x - xv(x)$ written as a function of $v = v(x)$. By the assumption that $xv(x)$ is strictly convex we have that there is a unique maximum obtained at a large value of v when β is large. Indeed the value of v that maximizes this exponent is found by differentiation to be determined by the equation

$$w(V(v)) = \beta + M \text{ or } v + (V/V')(v) = \beta + M. \tag{3.12}$$

The equation (3.12) for v may also be written as $w(x) = \beta + M$. Now we choose β implicitly by specifying the value of x or $v = v(x)$ that must correspond to that choice of β through (3.12) as: $v = v(x_\beta)$ for $x_\beta := U(\ln(2d))$. This therefore determines $\beta := w(x_\beta) - M = w(U(\ln(2d)) - M$. Note that this value of β tends to ∞ with d tending to ∞, since in particular by (3.6) $w(x) \ge c + v(x)$ for all large x and a constant $c > 0$. Since the solution for v in (3.12) is clearly in the interval $[v(x_0), 2\beta]$ for large β (again by $w(x) \ge c + v(x)$ for large x) we have solved the maximization problem for the upper bound of I in (3.11) with the given choice of $\beta = w(U(\ln(2d)) - M$ with large d and the unique critical value $v = v(U(\ln(2d))$. Since by (3.12) we have $\beta + M - v(x_\beta) = (V/V')(v(x_\beta))$, we find that the value of the maximum exponent is: $(V^2/V')(v) = x_\beta^2 v'(x_\beta) = U(w(U) - v(U)) = U^2 v'(U)$, for $U = U(\ln(2d))$. We thus have that

$$I \le C_0 w(x) V(2w(x)) \exp(x^2 v'(x)), \text{ for } x = U(\ln(2d)).$$

We need finally to estimate $w(x)V(2w(x))$ to handle the term before the exponential in this upper bound for I. However we can do this using (3.6) as follows. First we note the following observation.

Observation. If $f(x) \nearrow \infty$ and $g(x) \nearrow \infty$ and if f and g are differentiable with $f'(x), g'(x) > 0$, for all large x, and finally, if $f'(x) = o(1)g'(x)$ as $x \to \infty$, then $f(x) = o(g(x))$, as $x \to \infty$. Indeed this can easily be shown by a proof by contradiction.

Now we set up two functions f and g for this context. Since by (3.6), we have $1/(yv'(y)) \le C$ for a constant $C > 0$ and all large y, we have that $(V'/V)(v(y)) = 1/(yv'(y)) \le C$. Thus, since $v(y)$ stands for any large value of the argument we have that $(V'/V)(Aw(x)) \le C$, for all large x, and any constant $A > 0$. Now multiply both sides of this last relation by $Aw'(x)$ where we recall that $w'(x) = (xv(x))''$. Since by (3.4) $w'(x) > 0$, we therefore have: $(V'/V)(Aw(x)) \times Aw'(x) \le ACw'(x) = o(1)xw'(x)$. But by a straightforward computation $xw'(x) = (x^2v'(x))'$, and of course $(V'/V)(Aw(x)) \times Aw'(x) = (\ln(V(Aw(x))))'$. Hence by our observation above applied with $f(x) := \ln(V(Aw(x)))$ and $g(x) := x^2v'(x)$, we have that $\ln(V(Aw(x))) = o(x^2v'(x))$. Alternatively we have shown that V can grow at most exponentially, so $\ln(V(w(x))) << w(x)$, but since $xw'(x) = (x^2v'(x))'$, we have that $w'(x) = o(1)(x^2v'(x))'$. So by the observation, $w(x) = o(x^2v'(x))$, and we end up with the same conclusion. Thus because as noted already $x^2v'(x) \nearrow \infty$ as $x \to \infty$, we have by our estimation of I in (3.11) and $II = o(1)$ above that indeed

$\lambda_+(\beta_d) = \ln(I + II) \le x_\beta^2 v'(x_\beta)(1 + o(1))$ for $x_\beta = U(\ln(2d))$ and $\beta_d = w(x_\beta) - M$. $\square$

Proof of Theorem 1. The proof of Theorem 1 now follows by Lemmas 1 and 2 and the relations $\lambda_+(\beta) \ge \nu_d \beta - \ln(2d)$ and $\nu_d \ge m_d$ mentioned above. Thus, writing U in place of $U(\ln(2d))$ and $u(U)$ in place of $\ln(2d)$ for the nice exponent $u = xv(x)$, we have by Lemma 2 that

$$U^2 v'(U)(1 + o(1)) \ge (w(U) - M)\nu_d - \ln(2d), \text{ or, } (u(U) + U^2 v'(U))(1 + o(1)) \ge w(U)(1 + o(1))\nu_d.$$

But now since $u(U) + U^2 v'(U) = Uw(U)$, we have shown that $\nu_d \le U \cdot (1 + o(1))$. Since by Lemma 1 we also have that $\nu_d \ge U \cdot (1 + o(1))$, the proof is complete. $\square$

We illustrate Theorem 1 in the unit Poisson case. By Stirling's formula $\ln(x!) = c + o(1) + (x + 1/2)\ln(x) - x$, we find that the Poisson case is represented by $v(x) = \ln(x) - 1$ with $\delta(x) = O(\ln(x))$. The growth condition (3.6) is just satisfied: $xv'(x) = 1$, and the regularity condition (3.7) is easily satisfied since here $(xv(x))'^2/(xv(x))'' \sim x\ln(x)^2$. We have that U is the inverse of $xv(x) \sim x\ln(x)$, or $U(u) \sim u/\ln(u)$. Therefore in the Poisson case

$$\nu_d \sim \ln(2d)/\ln\ln(2d), \text{ as } d \to \infty.$$

Finally, we apply Theorem 1 to the case when the distribution $F(x)$ of η is a mixture: $F(x) = pF_0(x) + qF_1(x)$ for $F_i(x) = 1 - \exp(-u_i(x))$, $i = 0, 1$. Assume that u_0 satisfies the conditions of Theorem 1, and that $-u_1(x) + u_0(x) \le Mx$ for all large x and some constant $M > 0$. Then one easily calculates that the distribution $F(x)$ continues to have an exponent $-u_0(x)$ with a different error term $\delta(x)$ in (3.4). Therefore the last passage time constant for the mixture F is asymptotically the same as that for the distribution F_0. Thus the time constant is ruled by the distribution of the mixture with the heavier upper tail.

References

[1] I. Ben-Ari (2007), Large deviations for partition functions of directed polymers and some other models in an IID field, arXiV:0704.3758v1 [math.PR] 27 Apr 2007.

[2] E. Bolthausen (1989), A note on the diffusion of directed polymers in a random environment, *Commun. Math. Phys.* **123**, 529-534.

[3] P. Carmona and Y. Hu (2002), On the partition function of a directed polymer in a random environment, *Probab. Theory Relat. Fields* **124**, 431-457.

[4] F. Comets, T. Shiga, and N. Yoshida (2003), Directed polymers in random environment: path localization and strong disorder, *Bernoulli* **9** (4), 705-723.

[5] F. Comets, T. Shiga, and N. Yoshida (2004), Probabilistic analysis of directed polymers in a random environment: a review, in "Stochastic analysis on large scale interacting systems", *Adv Stud.Pure Math.* **39**, Math Soc. Japan, Tokyo, pp. 115-142.

[6] F. Comets and N. Yoshida (2006), Directed polymers in random environment are diffusive at weak disorder, *Ann. Probab.* **34** No. 5, 1746-1770. arXiv:math.PR/0411223 v2.

[7] F. Comets and V. Vargas (2005), Majorizing multiplicative cascades for directed polymers in random media, arXiv:math.PR/0510525.

[8] J.M. Hammersley and D.J.A. Welsh (1965), First passage percolation, subadditive processes, stochastic networks, generalized renewal theory. *Bernoulli-Bayes-Laplace Anniversary Volume*, J. Neyman and L.E. LeCam, eds. Springer-Verlag, Berlin.

[9] J.Z. Imbrie and T. Spencer (1988), Diffusion of directed polymers in a random environment, *J. Statist. Phys.*, **52** no. 3-4, 609-626.

[10] K. Johansson (2000), Shape fluctuations and random matrices, *Comm. Math. Phys.* **209**, 437-476.

[11] H. Kesten (1986), *Aspects of First Passage Percolation. Lecture Notes in Math.* **1180**, 125-264. Springer, Berlin.

[12] S. Lee (1997), The power laws of M and N in greedy lattice animals, *Stoch. Process. Appl.* **69**, 275-28

[13] J.B. Martin (2004), Limiting shape for directed percolation models, *Ann. Probab.* **32** No. 4, 2908-2937.

[14] J.B.Martin (2006), Last-passage percolation with general weight distribution, *Markov Processes and Related Fields*, **12**, 273-299.

[15] G.J. Morrow (2007), A directed polymer approach to once-oriented first passage site percolation in high dimensions, Preprint. http://www.uccs.edu/gmorrow.

[16] C.M. Newman and M.S.T. Piza (1995), Divergence of shape fluctuations in two dimensions, *Ann. Probab.* **23**, No. 3, 977-1005.

[17] R. Song and X.Y. Zhou (1996), A remark on diffusion of directed polymers in random environments, *J. Statist. Phys.* **85** Nos. 1/2, 277-289.

[18] R.T. Smythe and J.C. Wierman (1978), *First-Passage Percolation on the Square Lattice. Lectures Notes in Math.* **671**. Springer-Verlag, New York.

[19] M. Talagrand (2003), *Spin Glasses: A Challenge for Mathematicians, cavity and mean field models*, Springer, New York.

Dept. Mathematics, University of Colorado, Colorado Springs, CO 80933-7150.
gmorrow@uccs.edu