ON THE DEPENDENCE COEFFICIENTS ASSOCIATED WITH THREE MIXING CONDITIONS FOR RANDOM FIELDS

Dedicated to the memory of Walter Philipp

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Abstract. In connection with the ρ-mixing, ρ'-mixing, and ρ^*-mixing conditions for strictly stationary random fields indexed by an integer lattice of dimension at least 2, the simultaneous behavior of the dependence coefficients can be essentially arbitrary, subject to certain unavoidable elementary inequalities.

1. Introduction

For various special classes of stochastic sequences, there has been a lot of interest in the question of what "strong mixing conditions" they might satisfy and with what "mixing rates." For a well known "continued fraction" process and related processes, this question has been investigated by Gordin [19], Philipp [31] and others, and a thorough treatment is given in the book by Iosifescu and Kraaikamp [24]. For this question for strictly stationary Markov chains, see e.g. [34, Chapter 7] or [25, Theorem 4]. For this question for stationary Gaussian sequences, see e.g. [23]. For this question for ARMA processes, see e.g. [15]. Some information on this question for strictly stationary Markov chains and for stationary Gaussian sequences is also given in [7].

Of course most of the development of limit theory under strong mixing conditions in the literature is not restricted to such special classes of random sequences (though much of it *motivated* by such classes). In this more general development, there has been a very broad spectrum of "mixing rate" assumptions, ranging from exponential (or even faster), to polynomial, and even to logarithmic (starting with [22] and [2, Theorem 4]). In some limit theorems, the mixing rate can even be arbitrarily slow. An ongoing pertinent question has been what are the possible mixing rates for strong mixing conditions for (say) strictly stationary random sequences.

For several different strong mixing conditions for strictly stationary random sequences, Kesten and O'Brien [25] showed with some classes of examples that the mixing rates could be essentially arbitrary, and in particular, arbitrarily slow. Their examples provided a valuable service in that they "separated from each other" various mixing assumptions that had been used by many researchers in limit theory for mixing sequences. Further results in the spirit of Kesten and O'Brien can be found in [7, V3, Chapter 26] and the references cited there.

As another example, in connection with Eberlein's [17] version of the "very weak Bernoulli" condition for strictly stationary sequences of random variables taking their values on quite general metric spaces, Dehling, Denker, and Philipp [12] showed that the "mixing rate" cannot be $o(1/n)$ except in the trivial case of i.i.d. sequences. The author [3] showed with some examples that the "mixing rate" can be $O(1/n)$ or a fairly arbitrary slower rate, even when the random variables are real-valued and the sequence does not satisfy Rosenblatt's [32] strong mixing ("α-mixing") condition. For more on this, see e.g. [7, V1, sections 13.20-13.23] and the references cited there.

The question of possible mixing rates also arises in the context of random fields. Research pertinent to this question goes back at least to Dobrushin [14]; and further results can be found in [9, p. 13, Theorem 2.1], in [15], [5], and in [7, V3, Chapter 29] and the references cited there.

The purpose of this note is to address this question, for strictly stationary random fields, simultaneously with respect to three strong mixing conditions that are based on the "maximal correlation" coefficient. Those conditions will be formally defined in Definition 1.6 below, after some basic notations and a review of a broad class of dependence coefficients and the connections between them. The main result will be given in Theorem 1.9.

Then that result will be motivated by various known central limit theorems for strictly stationary random fields. Definitions and elementary observations will often both be included together under the heading "Notations."

Notations 1.1. (a) For any two nonempty sets $G, H \subset \mathbf{Z}$, define the "distance" between them by

$$\mathrm{dist}(G, H) := \inf_{g \in G, h \in H} |g - h|. \tag{1.1}$$

(Of course this quantity is 0 if G and H have an element in common.)

(b) For each positive integer n, let $\mathcal{E}(n)$ denote the family of all ordered pairs (G, H) of nonempty subsets of $\mathbf{Z}$ such that

$$\mathrm{dist}(G, H) \geq n.$$

(Of course the sets G and H can be "interlaced"; each one can have elements between ones in the other set.)

Notations 1.2. (a) Suppose d is a positive integer. Any given element $k \in \mathbf{Z}^d$ will be represented by

$$k := (k_1, k_2, \ldots, k_d). \tag{1.2}$$

(b) For each $k \in \mathbf{Z}^d$ and each $p \in [1, \infty)$, define the usual "p-norm" of k by

$$\|k\|_p := (|k_1|^p + |k_2|^p + \ldots + |k_d|^p)^{1/p}.$$

For each $k \in \mathbf{Z}^d$, define the usual "∞-norm" of k by

$$\|k\|_\infty := \max\{|k_1|, |k_2|, \ldots, |k_d|\}.$$

(c) It is well known and elementary that for a given $k \in \mathbf{Z}^d$,

$$\text{if} \ \ 1 \leq p < q \leq \infty \ \ \text{then} \ \ \|k\|_q \leq \|k\|_p. \tag{1.3}$$

(See e.g. [7, V1, Appendix, Lemma A406] for the case $q < \infty$.)

(d) The set $[1, \infty) \cup \{\infty\}$ will be denoted simply by $[1, \infty]$. For any $p \in [1, \infty]$ and any two nonempty sets $S, T \subset \mathbf{Z}^d$, define the "$p$-distance" between them by

$$\mathrm{dist}_p(S, T) := \min_{s \in S, t \in T} \|s - t\|_p. \tag{1.4}$$

(Of course this quantity is 0 if S and T have an element in common.)

(e) By (c), for any two nonempty sets $S, T \subset \mathbf{Z}^d$,

$$\text{if} \ \ 1 \leq p < q \leq \infty \ \ \text{then} \ \ \mathrm{dist}_q(S, T) \leq \mathrm{dist}_p(S, T). \tag{1.5}$$

(f) Of course in the case $d = 1$, one has that if $p \in [1, \infty]$, then (i) $\|k\|_p = |k|$ for any $k \in \mathbf{Z}$, and (ii) $\mathrm{dist}_p(S,T) = \mathrm{dist}(S,T)$ (see (1.1)) for any two nonempty sets $S, T \subset \mathbf{Z}$.

Notations 1.3. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

For any two σ-fields $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$, define the following two measures of dependence:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|; \tag{1.6}$$

and

$$\rho(\mathcal{A},\mathcal{B}) := \sup |\mathrm{Corr}(f,g)| \tag{1.7}$$

where this latter supremum is taken over all pairs of square-integrable random variables f and g such that f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable.

It is well known and elementary (see e.g. [7, V1, Proposition 3.11(b)]) that for any two σ-fields $\mathcal{A}$ and $\mathcal{B}$,

$$0 \le 4\alpha(\mathcal{A},\mathcal{B}) \le \rho(\mathcal{A},\mathcal{B}) \le 1. \tag{1.8}$$

(Of course, if $\mathcal{A}$ and $\mathcal{B}$ are independent, then $\alpha(\mathcal{A},\mathcal{B}) = 0$ and $\rho(\mathcal{A},\mathcal{B}) = 0$; and vice versa.)

Notations 1.4. Suppose d is a positive integer. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a strictly stationary random field on our given probability space $(\Omega, \mathcal{F}, P)$.

(a) For each positive integer n, define (see (1.2)) the following dependence coefficient:

$$\alpha(X,n) \ := \sup_{u \in \{1,2,\dots,d\}} \alpha(\sigma(X_k, k_u \le 0), \sigma(X_k, k_u \ge n)). \tag{1.9}$$

Here and below, the notation $\sigma(\dots)$ means the σ-field of events generated by $(\dots)$. The notation $\sigma(X_k, k_u \le 0)$ means of course the σ-field of events generated by the random variables X_k for $k := (k_1, k_2, \dots, k_d) \in \mathbf{Z}^d$ such that $k_u \le 0$. The obvious analogous convention holds for the notation $\sigma(X_k, k_u \ge n)$ in (1.9) as well as in similar notations in (1.10) and (1.11) below.

Trivially, $\alpha(X,1) \ge \alpha(X,2) \ge \alpha(X,3) \ge \dots$. The analogous comment applies to the other dependence coefficients defined below.

(b) For each positive integer n, define the following two dependence coefficients:

$$\rho(X,n) \ := \sup_{u \in \{1,2,\dots,d\}} \rho(\sigma(X_k, k_u \le 0), \sigma(X_k, k_u \ge n)); \tag{1.10}$$

and (see Notations 1.1(b))

$$\rho'(X,n) \ := \sup_{u \in \{1,2,\dots,d\},\ (G,H) \in \mathcal{E}(n)} \rho(\sigma(X_k, k_u \in G), \sigma(X_k, k_u \in H)). \tag{1.11}$$

(c) For each $p \in [1, \infty]$ and each positive integer n, define the dependence coefficient

$$\rho_p^*(X,n) \ := \ \sup \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \tag{1.12}$$

where the supremum is taken over all pairs of nonempty sets $S, T \subset \mathbf{Z}^d$ such that

$$\mathrm{dist}_p(S,T) \geq n.$$

Remark 1.5. Again suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a strictly stationary random field.

(a) For a given $n \in \mathbf{N}$ (the set of all positive integers), one has by (1.5) and a trivial argument that

$$\text{if} \quad 1 \leq p < q \leq \infty \quad \text{then} \quad \rho_q^*(X,n) \leq \rho_p^*(X,n). \tag{1.13}$$

Also, for a given $k \in \mathbf{Z}^d$, $\|k\|_1 \leq d \cdot \|k\|_\infty$. Hence for a given $n \in \mathbf{N}$, if S and T are nonempty subsets of $\mathbf{Z}^d$ such that $\mathrm{dist}_1(S,T) \geq dn$, then $\mathrm{dist}_\infty(S,T) \geq n$. Hence for a given $n \in \mathbf{N}$, $\rho_1^*(X, dn) \leq \rho_\infty^*(X,n)$. From that and (1.13) one now has that for a given positive integer n,

$$\forall p \in [1,\infty], \quad \rho_1^*(X,dn) \leq \rho_\infty^*(X,n) \leq \rho_p^*(X,n) \leq \rho_1^*(X,n). \tag{1.14}$$

Hence for the different values of $p \in [1,\infty]$, the conditions $\rho_p^*(X,n) \to 0$ (as $n \to \infty$) are equivalent, and if they hold then the corresponding mixing rates are in a certain sense "of similar order."

(b) Also, by a trivial argument, for each positive integer n,

$$\rho(X,n) \leq \rho'(X,n) \leq \rho_\infty^*(X,n). \tag{1.15}$$

(c) If $d \geq 2$, then by strict stationarity and [7, V3, Theorem 29.12], the conditions $\alpha(X,n) \to 0$ (as $n \to \infty$) and $\rho(X,n) \to 0$ are equivalent, and if they hold then

$$\forall n \in \mathbf{N}, \quad \alpha(X,n) \leq \rho(X,n) \leq 2\pi\alpha(X,n). \tag{1.16}$$

(d) Suppose one defines $\alpha'(X,n)$ exactly analogously to (1.11) but with the ρ in the right side of that equation replaced by α. Then by strict stationarity and [7, V3, Theorem 29.12], regardless of whether $d = 1$ or $d \geq 2$, the conditions $\alpha'(X,n) \to 0$ and $\rho'(X,n) \to 0$ are equivalent, and if they hold then

$$\forall n \in \mathbf{N}, \quad \alpha'(X,n) \leq \rho'(X,n) \leq 2\pi\alpha'(X,n). \tag{1.17}$$

(e) Suppose one defines $\alpha_p^*(n)$ exactly analogously to (1.12) but with the ρ in the right side of that equation replaced by α. Then by strict stationarity and [7, V3, Theorem 29.12], regardless of whether $d = 1$ or $d \geq 2$, the conditions $\alpha_2^*(X,n) \to 0$ (as $n \to \infty$) and $\rho_2^*(X,n) \to 0$ are equivalent; and if they hold then

$$\forall n \in \mathbf{N}, \quad \alpha_2^*(X,n) \leq \rho_2^*(X,n) \leq 2\pi\alpha_2^*(X,n). \tag{1.18}$$

Also, comment (a) above (and in particular its last sentence) holds verbatim with $\rho_p^*(X,n)$ replaced by $\alpha_p^*(X,n)$ for each $p \in [1,\infty]$. Hence by (1.18), the various conditions $\rho_p^*(X,n) \to 0$ and $\alpha_p^*(X,n) \to 0$, for $p \in [1,\infty]$, are all equivalent of each other, and if they hold then the various corresponding mixing rates are in a certain sense "of similar order." (Also, adapting the argument in [7, V3, proof of Theorem 29.12(I(A))], one can obtain analogs of (1.18) for other values of p besides 2; but that will not be important here.)

(f) Because of comments (c), (d), and (e) above, we shall not deal further explicitly with the dependence coefficients $\alpha'(X,n)$ or $\alpha_p^*(X,n)$, and we shall deal with $\alpha(X,n)$ only when $d=1$.

(g) Refer to (e). In the special case where $X := (X_k, k \in \mathbf{Z}^d)$ is a stationary *Gaussian* random field with a continuous positive spectral density function, one has that (i) $\rho_1^*(X,n) \to 0$ as $n \to \infty$, by [35, p. 73, Theorem 7] (generalizing an argument from [26] involving $\rho(X,n)$ for the case $d=1$), and (ii) $\rho_1^*(X,1) < 1$, by an argument that goes back to calculations (for the case $d=1$) of Moore [29]. (Proofs of both (i) and (ii) can also be found in [7, V3, Theorems 25.47 and 28.38].

Definition 1.6. A given strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ (whether $d=1$ or $d \geq 2$) is said to be

(i) "ρ-mixing" if $\rho(X,n) \to 0$ as $n \to \infty$
(ii) "ρ'-mixing" if $\rho'(X,n) \to 0$ as $n \to \infty$,
(iii) "ρ^*-mixing" if $\rho_p^*(X,n) \to 0$ as $n \to \infty$, for any (hence all) $p \in [1,\infty]$.

(Recall Remark 1.5(a)). Of course by (1.15), ρ^*-mixing implies ρ'-mixing, and ρ'-mixing implies ρ-mixing.

Remark 1.7. Let us digress for a moment to consider the case $d=1$. Suppose $X := (X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of random variables. In this case $d=1$, the following observations hold:

(a) For every positive integer n and every $p \in [1,\infty]$, one trivially has that $\rho_p^*(X,n) = \rho'(X,n)$.

(b) The condition $\alpha(X,n) \to 0$ (as $n \to \infty$) is simply the "strong mixing" (or "α-mixing") condition introduced by Rosenblatt [32].

(c) The ρ-mixing condition $\rho(X,n) \to 0$ was introduced by Kolmogorov and Rozanov [26] in their study of stationary Gaussian sequences. (The maximal correlation coefficient $\rho(\mathcal{A},\mathcal{B})$ itself had been studied earlier by other researchers, starting with Hirschfeld [21], in statistical contexts that did not involve "stochastic processes.")

(d) The ρ-mixing condition is strictly stronger than strong mixing (α-mixing). See e.g. the examples in Davydov [11] or [7, V1, Example 7.11]. (Note the contrast with Remark 1.5(c) above, involving the case $d \geq 2$.)

(e) Suppose $(q_n, n \in \mathbf{N})$, $(r_n, n \in \mathbf{N})$, and $(s_n, n \in \mathbf{N})$ are each a *nonincreasing* sequence of numbers in $(0,1]$, and that for each positive integer n,

$$0 < 4q_n \leq r_n \leq s_n \leq 1.$$

Suppose also that $(t_n, n \in \mathbf{N})$ is a sequence of positive numbers. Under these conditions, there exists a strictly stationary random sequence $X := (X_k, k \in \mathbf{Z})$ such that (i) the marginal distribution of X_0 is purely nonatomic and (ii) for each $n \geq 1$ (and each $p \in [1, \infty]$ — see (a) above),

$$\begin{aligned} &q_n \leq \alpha(X, n) \leq q_n + t_n, \\ &\rho(X, n) = r_n, \quad \text{and} \\ &\rho_p^*(X, n) = \rho'(X, n) = s_n. \end{aligned}$$

(See [7, V3, Theorem 26.8 and its subsequent Note 3]. The word "atomic" in that Note 3 should be "nonatomic.")

Section 1.8. Now let us turn our attention to random fields indexed by an integer lattice of dimension at least two. Suppose d is an integer such that $d \geq 2$. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a strictly stationary random field.

(a) Let $\mathbf{0} := (0, 0, \ldots, 0)$ denote the origin in $\mathbf{Z}^d$.

(b) For each $u \in \{1, 2, \ldots, d\}$, let $\mathbf{e}(u)$ denote the uth coordinate unit vector in $\mathbf{Z}^d$:

$$\mathbf{e}(u) := (0, \ldots, 0, 1, 0, \ldots, 0)$$

where the 1 is the uth coordinate. Thus for example, $7\mathbf{e}(3) = (0, 0, 7, 0, 0, \ldots, 0)$. (Similar notations will arise frequently in what follows.)

(c) For a given positive integer n and a given $u \in \{1, 2, \ldots, d\}$,

$$\rho(X, n) \ \geq \ \rho(\sigma(X_k, k_u \leq 0), \sigma(X_k, k_u \geq n)) \ \geq \ \rho(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)})), \tag{1.19}$$

and (see Notation 1.1(b))

$$\begin{aligned} \rho'(X, n) \ &\geq \sup_{(G,H) \in \mathcal{E}(n)} \rho(\sigma(X_k, k_u \in G), \sigma(X_k, k_u \in H)) \\ &\geq \ \rho(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)}, X_{-n\mathbf{e}(u)})). \end{aligned} \tag{1.20}$$

(d) For a given positive integer n and a given pair of distinct indices $u, v \in \{1, 2, \ldots, d\}$,

$$\rho(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)}, X_{n\mathbf{e}(v)})) \ \leq \ \rho_\infty^*(X, n). \tag{1.21}$$

Now let us state the main result of this paper. It gives for $d \geq 2$ a "rough analog" (recall Remark 1.5(c)) of the result for $d = 1$ mentioned in Section 1.7(e) above.

Theorem 1.9. *Suppose $(a_n, n \in \mathbf{N})$, $(b_n, n \in \mathbf{N})$, and $(c_n, n \in \mathbf{N})$ are each a nonincreasing sequence of numbers in $(0, 1]$, and suppose these numbers satisfy*

$$\forall n \in \mathbf{N}, \quad 0 < a_n \leq b_n \leq c_n \leq 1. \tag{1.22}$$

Suppose d is an integer such that $d \geq 2$.

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) the (marginal) distribution of $X_{\mathbf{0}}$ is purely nonatomic and (ii) the following equalities (1.23)–(1.28) hold:

$$\forall n \in \mathbf{N}, \quad \rho(X, n) = a_n; \tag{1.23}$$

$$\forall n \in \mathbf{N}, \ \forall u \in \{1, 2, \dots, d\}, \quad \rho(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)})) = 4\alpha(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)})) = a_n; \tag{1.24}$$

$$\forall n \in \mathbf{N}, \quad \rho'(X, n) = b_n; \tag{1.25}$$

$$\forall n \in \mathbf{N}, \ \forall u \in \{1, 2, \dots, d\}, \quad \rho(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)}, X_{-n\mathbf{e}(u)})) = b_n; \tag{1.26}$$

$$\forall n \in \mathbf{N}, \quad \rho_1^*(X, n) = c_n; \tag{1.27}$$

and for every $n \in \mathbf{N}$ and every pair of distinct indices $u, v \in \{1, 2, \dots, d\}$,

$$\rho(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)}, X_{n\mathbf{e}(v)})) = c_n. \tag{1.28}$$

Remark 1.10. (a) In Theorem 1.9, a_n is not assumed to converge to 0. The random field X need not be ρ-mixing.

(b) By Remark 1.8(c)(d) and eq. (1.14), one has that eqs. (1.23)–(1.28) implicitly contain some further information regarding the random field X in Theorem 1.9:

$$\forall n \in \mathbf{N}, \ \forall u \in \{1, 2, \dots, d\}, \quad \rho(\sigma(X_k, k_u \le 0), \sigma(X_k, k_u \ge n) = a_n; \tag{1.29}$$

$$\forall n \in \mathbf{N}, \ \forall u \in \{1, 2, \dots, d\}, \quad \sup_{(G,H) \in \mathcal{E}(n)} \rho(\sigma(X_k, k_u \in G), \sigma(X_k, k_u \in H)) = b_n; \tag{1.30}$$

$$\forall n \in \mathbf{N}, \ \forall p \in [1, \infty], \quad \rho_p^*(X, n) = c_n. \tag{1.31}$$

Eqs. (1.24), (1.26), and (1.28) are intended to show that for the random field X described in Theorem 1.9, the particular values of the dependence coefficients given in (1.23), (1.25), and (1.27) are in a certain sense "pervasive."

(For the strictly stationary random *sequence* in the context of the theorem described in Remark 1.7(e), with a careful examination of the proof of that theorem, one obtains extra information of a spirit somewhat similar to (1.24) and (1.26). We shall not bother with that further here.)

(c) In a straightforward way, one can extend Theorem 1.9 to a form in which for some integer $N \ge 2$, eq. (1.22) (including $0 < a_n$) holds for all $n \in \{1, 2, \dots, N\}$ and $a_n = b_n = c_n = 0$ for all $n \ge N + 1$. (In key places in the argument in Section 5 below, one simply uses only finitely many "building block" random fields instead of countably many of them.) Other "extensions with zeros" are trickier, and will not be treated here beyond the following brief comments: By an elementary argument, for a given $n \in \mathbf{N}$, if $\rho(X, n) = 0$ then $\rho'(X, n) = 0$. (See e.g. [7, V3, Proposition 29.5].) Also, by an elementary argument, if $\rho(X, 1) = 0$ then $\rho^*(X, 1) = 0$ (i.e. the random field X is i.i.d.). For a given integer $d \ge 2$, the author [6] constructed a (nondegenerate) strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ that satisfies $\rho(X, 2) = 0$ (and hence $\rho'(X, 2) = 0$) and $\rho^*(X, n) = 1$ for all $n \in \mathbf{N}$. (In that example, the "tail σ-field" of X contains, modulo null-sets, the entire history of X.)

(d) Since the (marginal) distribution of $X_{\mathbf{0}}$ is purely nonatomic, one can trivially obtain any moment or "tail" condition on that marginal distribution (for example, $EX_{\mathbf{0}}^2 < \infty$ together with $E|X_{\mathbf{0}}|^{2+\delta} = \infty$ for all $\delta > 0$), while retaining all properties stated in Theorem 1.9, simply by applying to the X_k's an appropriate one-to-one bimeasurable function from $\mathbf{R}$ onto $\mathbf{R}$.

(e) Up to now, the random variables have been tacitly understood to be real-valued. However, it is well known that (for example) any (nontrivial) separable Banach space is bimeasurably isomorphic to the real number line $\mathbf{R}$. (For a more general theorem, see e.g. [16, Theorem 13.1.1].) The definitions and observations in Sections 1.1-1.8 carry over trivially to (say) Banach-space-valued random variables. Thus again using the fact that in Theorem 1.9, the marginal distribution is purely nonatomic, Theorem 1.9 can be extended directly to provide meaningful, nontrivial examples of strictly stationary random fields $X := (X_k, k \in \mathbf{Z}^d)$ (for $d \geq 2$) with prescribed mixing rates, with the random variables taking their values in a separable Banach space, with appropriate "moment" qualities. As motivation, in Remark 1.13 below, we shall look at a central limit theorem in [38] involving ρ'-mixing random fields with the random variables talking their values in a separable real Hilbert space.

(Similarly, the theorem mentioned in Remark 1.7(e) accomplishes a task analogous to that of Theorem 1.9 for strictly stationary *sequences* of random variables taking their values in a separable Banach space. As motivation, consider the limit theorems in [13] and [27] involving mixing sequences of Hilbert-space-valued or Banach-space-valued random variables.)

(f) In the special case where $a_n = b_n = c_n$ for each $n \in \mathbf{N}$, the general spirit of Theorem 1.9 had already been shown in [5, Theorem 3].

(g) In (1.24), a nice piece of "extra" information (the equality $\alpha(\sigma(X_{\mathbf{0}}), \sigma(X_{n\mathbf{e}(u)})) = a_n/4$ given there) is obtained trivially in the course of the proof of Theorem 1.9 given below, with essentially no extra work. (Such is not the case in connection with either (1.26) or (1.28); we shall not bother further with that here.) In much of the central limit theory for strictly stationary random fields $X := (X_k, k \in \mathbf{Z}^d)$, there are assumptions like

$$\alpha(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \ \leq \ A \cdot (\operatorname{card} T) \cdot (\operatorname{dist}_2(S, T))^{-c}$$

(where A and c are fixed positive constants), for nonempty disjoint subsets $S, T \subset \mathbf{Z}^d$ such that the cardinality of T is finite. See e.g. [39] or [15]. (For such assumptions, involving in a critical way the (finite) cardinality of one or both index sets, analogs of Remark 1.5(c)(d)(e) do not hold; see e.g. [5, Theorems 1 and 2].) Because of (1.24), Theorem 1.9 shows (as did [5, Theorem 3]) that limit theorems under such assumptions will not yield as corollaries limit theorems involving (say) ρ-mixing with a "very slow" mixing rate. This is pertinent to Remarks 1.11–1.14 below.

Remark 1.11. Central limit theory for strictly stationary ρ-mixing random sequences ($d = 1$) apparently started with results of Rosenblatt [33][34] in the context of Markov chains, and was then promoted by Ibragimov [22] in the general (non-Markovian) context.

Starting with that paper of Ibragimov, for central limit theory for strictly stationary ρ-mixing sequences with just the assumption of finite second moment, the logarithmic mixing rate assumption

$$\sum_{n=0}^{\infty} \rho(X, 2^n) < \infty \tag{1.32}$$

became standard. In that context, that mixing rate assumption is essentially sharp; see [7, V3, Theorems 34.13 and 34.15].

For strictly stationary ρ-mixing random fields $X := (X_k, k \in \mathbf{Z}^d)$ for $d \geq 2$, that logarithmic mixing rate assumption (1.32) continues to be quite standard. See e.g. the paper of Gorodetskiĭ [20], involving the closely related context of random fields indexed by $\mathbf{R}^d$. Theorem 1.9 shows that for $d \geq 2$, there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) $\rho(X, n) = 1/(\log n)^{1.001}$ for all $n \geq 3$ (and hence (1.32) is satisfied) and (ii) $\rho_p^*(X, n) = \rho'(X, n) = 1$ for all $n \geq 1$ (and all $p \in [1, \infty]$). Thus central limit theorems for strictly stationary ρ-mixing random fields under that logarithmic mixing rate (1.32) cannot be derived from central limit theorems for ρ^*-mixing or even ρ'-mixing random fields.

Remark 1.12. Next let us consider the ρ^*-mixing condition. Strictly stationary ρ^*-mixing random sequences ($d = 1$) with an exponential mixing rate, were one context in which Charles Stein [37] developed what is now known as "Stein's method" for obtaining rates of convergence in central limit theorems.

Later, for strictly stationary ρ^*-mixing random fields $X := (X_k, k \in \mathbf{Z}^d)$ for $d \geq 2$, central limit theory was developed by Goldie and Greenwood [18] and others, and still later it was shown by the author [4] that in central limit theorems for strictly stationary ρ^*-mixing random fields $X := (X_k, k \in \mathbf{Z}^d)$ under just finite second moments, the "mixing rate" for ρ^*-mixing can be permitted to be arbitrarily slow. (Since then, the same has come to be understood with regard to ρ'-mixing; more on that in Remark 1.13 below.)

By Theorem 1.9, for $d \geq 2$, there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ such that $\rho(X, n) = \rho'(X, n) = \rho_1^*(X, n) = 1/(\log(\log n))$ for all $n \in \mathbf{N}$ sufficiently large. (For $d = 1$, a similar comment holds by the theorem alluded to in Remark 1.7(e).) Thus central limit theorems for strictly stationary ρ^*-mixing random fields (with no assumption on the mixing rate) cannot be derived from, for example, central limit theorems for ρ-mixing random fields with the mixing rate assumption (1.32).

Part of the motivation for Theorem 1.9 is that in more sophisticated limit theorems for strictly stationary ρ^*-mixing (or perhaps even just ρ'-mixing) random fields $X := (X_k, k \in \mathbf{Z}^d)$ with $d \geq 2$ — for example, rates of convergence in the central limit theorem (possibly with Stein's method), or perhaps some sort of almost sure invariance principle (à la Berkes and Morrow [1]) — the use of mixing rate assumptions would probably be necessary.

Remark 1.13. Now let us look at the ρ'-mixing condition, this time with Hilbert-space-valued random variables. Suppose H is a (nontrivial) separable real Hilbert space with norm $\|.\|_H$ and origin $\mathbf{0}_H$; and suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a strictly stationary random field whose random variables X_k take their values in H. A given element $L \in \mathbf{N}^d$ will be represented à la (1.2) by $L := (L_1, L_2, \dots, L_d)$. For a given $L \in \mathbf{N}^d$, define the "block sum"

$$S(X, L) \;:=\; \sum X_k \tag{1.33}$$

where the sum is taken over all $k := (k_1, k_2, \dots, k_d)$ such that $1 \le k_u \le L_u$ for each $u \in \{1, 2, \dots, d\}$.

Suppose further (in addition to strict stationarity) that $EX_{\mathbf{0}} = \mathbf{0}_H$ and $E\|X_{\mathbf{0}}\|_H^2 < \infty$, and that the random field X is ρ'-mixing. Under these assumptions, Cristina Tone [38] proved that there exists a (possibly degenerate) Gaussian H-valued random variable Y with $EY = \mathbf{0}_H$ and $E\|Y\|_H^2 < \infty$ such that

$$(L_1 \cdot L_2 \cdot \ldots \cdot L_d)^{-1/2} S(X, L) \Longrightarrow Y \quad \text{as} \quad \min\{L_1, L_2, \dots, L_d\} \to \infty.$$

Here $\Longrightarrow$ denotes convergence in distribution. In her result, for an arbitrary given orthonormal basis of H, Tone also gave an explicit (if slightly complicated) formula for the "$\mathbf{N} \times \mathbf{N}$ covariance matrix" of Y with respect to that basis. (As part of her argument, she showed that that "matrix" is symmetric and positive semi-definite and has finite trace).

(Tone [38] also used that result, with $H = \mathbf{R}^m$ for $m \in \mathbf{N}$, in the proof of a functional central limit theorem for empirical processes from some strictly stationary ρ'-mixing random fields $X := (X_k, k \in \mathbf{Z}^d)$.)

By Theorem 1.9, for any given integer $d \ge 2$, there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$, with the random variables taking their values in a nontrivial way in a given (nontrivial) separable real Hilbert space H, such that (i) $\rho_p^*(X, n) = 1$ for all $n \ge 1$ and all $p \in [1, \infty]$ (see (1.27) and (1.31)), and (ii) $\rho'(X, n) = \rho(X, n) = 1/(\log(\log n))$ for all $n \in \mathbf{N}$ sufficiently large. Thus for $d \ge 2$, the result of Tone mentioned above cannot be derived from other limit theorems for random fields involving either an assumption of $\rho_p^*(X, n) < 1$ (for some n) or ρ-mixing with the standard logarithmic mixing rate assumption (1.32).

Remark 1.14. Now suppose d and m are each a positive integer, and $X := (X_k, k \in \mathbf{Z}^d)$ is a strictly stationary random field with the random variables X_k taking their values in the finite-dimensional Euclidean space $\mathbf{R}^m$. The origin in $\mathbf{R}^m$ will be denoted $\mathbf{0}_m$. (The origin in $\mathbf{Z}^d$ will continue to be denoted simply $\mathbf{0}$.) Again we use the notation $L := (L_1, L_2, \dots, L_d)$ for elements of $L \in \mathbf{N}^d$, and also the notation $S(X, L)$ in (1.33) for the "block sums." Suppose also that $EX_{\mathbf{0}} = \mathbf{0}_m$, and that all of the coordinates of $X_{\mathbf{0}}$ have finite second moments. For each $L \in \mathbf{N}^d$, let Σ_L denote the covariance matrix of the "block sum" $S(X, L)$. (Of course Σ_L is symmetric and positive semi-definite.)

Suppose also that $\rho'(X, 1) < 1$ and $\alpha(X, n) \to 0$ as $n \to \infty$. Of course keep in mind Remark 1.7(a) for the case $d = 1$, and Remark 1.5(c) for the case $d \ge 2$. (Here ρ'-mixing is not assumed.)

Under these assumptions, Tone [38] showed that if the covariance matrix of $X_{\mathbf{0}}$ is nonsingular, then (i) for every $L \in \mathbf{N}^d$, the covariance matrix Σ_L is nonsingular, and

(ii) one has that

$$\Sigma_L^{-1/2} S(X, L) \Longrightarrow N(\mathbf{0}_m, I_m) \quad \text{as} \quad \|L\|_\infty \to \infty.$$

Here (a) $S(X, L)$ is represented as an $m \times 1$ "random column vector," (b) $\Sigma_L^{-1/2}$ is a well known symmetric, positive definite $m \times m$ matrix whose square is the inverse of the matrix Σ_L (which is nonsingular by conclusion (i) above), and (c) $\Sigma_L^{-1/2} S(X, L)$ is the matrix product — itself an $m \times 1$ "random column vector." The notation $N(\mathbf{0}_m, I_m)$ represents the centered normal distribution on $\mathbf{R}^m$ whose covariance matrix is the $m \times m$ identity matrix I_m.

For the case $m = 1$ (real-valued random variables) and general $d \in \mathbf{N}$, this theorem boils down to a central limit theorem of a more conventional type given in [7, V3, Corollary 29.33]. For the special case $d = m = 1$, it is due to Peligrad [30].

By Theorem 1.9, for any given integer $d \geq 2$, there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$, with the random variables taking their values in $\mathbf{R}^m$ in a nontrivial way, such that (i) $\rho_p^*(X, n) = 1$ for all $n \geq 1$ and all $p \in [1, \infty]$ (see (1.27) and (1.31)), (ii) $\rho'(X, n) = .997$ for all $n \geq 1$, and (iii) $\rho(X, n) = 1/(\log(\log n))$ for all $n \in \mathbf{N}$ sufficiently large. Thus for $d \geq 2$ the result of Tone mentioned in the second paragraph above cannot be derived from other limit theorems for random fields involving an assumption of $\rho_p^*(X, n) < 1$ (for some n), ρ'-mixing, and/or ρ-mixing with the standard logarithmic mixing rate assumption (1.32). (For $d = 1$, a comment of a similar spirit holds by Remark 1.7(e).)

Remark 1.15. More work is waiting to be done in sorting out, à la Theorem 1.9, dependence coefficients similar to the ones in that theorem. For example, for (say) strictly stationary random fields $X := (X_t, t \in \mathbf{R}^d)$ (note the index set), there is more flexibility in defining analogs of $\rho(X, n)$ (see e.g. [20]), $\rho'(X, n)$, and $\rho_p^*(X, n)$ — and thus extra complications in sorting out the connections between the various possible definitions.

For weakly (i.e. second order) stationary random fields $X := (X_k, k \in \mathbf{Z}^d)$ (with the random variables X_k being real- or complex-valued), there are well known "linear" analogs of the dependence coefficients $\rho(X, n)$, $\rho'(X, n)$, and $\rho_p^*(X, n)$ which involve just linear combinations of the X_k's. For those analogous "linear dependence coefficients," the picture is quite different from Theorem 1.9 in that a direct analog of that theorem *fails* to hold — as one can see from various results in [8] or [7, V3, Chapter 28]. (For example, by Theorem 1.9, (1.32) does not imply ρ'-mixing, but in contrast — a result of Sergey Utev — the "linear dependence" analog of (1.32) does imply the "linear dependence" analog of ρ'-mixing; see [8, Theorem 3] or [7, V3, Theorem 28.5].) For such "linear dependence coefficients," the connections between them are in some ways tied to their connections with properties (e.g. boundedness or continuity) of a spectral density function. See [8] or [7, V3, Chapter 28] or (for weakly stationary random sequences $(X_k, k \in \mathbf{Z})$) [23, Chapter 5].

For "linear dependence coefficients" for weakly stationary random fields $X := (X_t, t \in \mathbf{R}^d)$ (again note the index set, and again the random variables can be complex-valued), the picture becomes even a little more complicated. As motivation, consider the investigations of Curtis Miller [28] and Jason Shaw [36] on the question of possible sufficient conditions for such random fields to have a continuous spectral density function. A theorem in Miller [28] gives that under mild technical extra assumptions, ρ^*-mixing (suitably formulated for the index set $\mathbf{R}^d$) is sufficient. His argument really involved only the corresponding "linear dependence" analog of ρ^*-mixing. Shaw [36] extended the result by showing that its assumption of ρ^*-mixing can be replaced by two other particular "linear dependence" assumptions that are each related to but weaker than a natural "linear dependence" analog of ρ'-mixing. Building on the work in [28] and [36], and in other references such as [23, Chapter 6] (involving weakly stationary random processes $(X_t, t \in \mathbf{R})$), quite a bit of work remains to be done in sorting out, for such random fields, the "linear dependence coefficients," the connections between them, and their connections with properties of spectral density.

The proof of Theorem 1.9 will be given in Sections 2–5. Sections 2, 3, and 4 will each develop a key "building block" random field for the final argument given in Section 5.

2. Part 1 of proof of Theorem 1.9

The main goal of Section 2 here is Lemma 2.5 below. It will provide one of the key "building block" random fields that will be used (in Section 5) in the construction of the random field X for Theorem 1.9. Lemma 2.5 will be achieved through an intermediate step, in Lemma 2.4. First, some key background material will be given in Lemmas 2.1 and 2.2 and Definition 2.3.

Lemma 2.1 (Csáki and Fischer). *Suppose $\mathcal{A}_n,\ n \in \mathbf{N}$ and $\mathcal{B}_n,\ n \in \mathbf{N}$ are σ-fields, and the σ-fields $\mathcal{A}_n \vee \mathcal{B}_n,\ n \in \mathbf{N}$ are independent. Then*

$$\rho\Big(\bigvee_{n\in\mathbf{N}} \mathcal{A}_n,\ \bigvee_{n\in\mathbf{N}} \mathcal{B}_n \Big) = \sup_{n\in\mathbf{N}} \rho(\mathcal{A}_n, \mathcal{B}_n)\ .$$

This theorem is due to Csáki and Fischer [10, Theorem 6.2]. A generously detailed proof (essentially the argument given by Witsenhausen [40], with more details) is given in [7, V1, Theorem 6.1].

The next lemma is [7, V3, Lemma 26.11].

Lemma 2.2. *Suppose $0 < \varepsilon \le r \le 1$. Then there exists a random vector $V := (V_1, V_2, V_3)$ with the following properties:*

$$\rho(\sigma(V_1),\, \sigma(V_2, V_3)) \ \le\ \varepsilon; \tag{2.1}$$

$$\rho(\sigma(V_1, V_2),\, \sigma(V_3)) \ \le\ \varepsilon; \tag{2.2}$$

$$\rho(\sigma(V_1, V_3),\, \sigma(V_2)) \ =\ r; \quad \text{and} \tag{2.3}$$

$$\text{the random variables } V_1 \text{ and } V_3 \text{ are independent.} \tag{2.4}$$

Definition 2.3. The following three notations will be useful.

(a) For a given random variable or random vector W, its distribution (on $\mathbf{R}$ or $\mathbf{R}^n$ for the appropriate positive integer n) will be denoted $\mathcal{L}(W)$.

(b) For each positive integer n, let $\phi_n : \mathbf{R}^n \to \mathbf{R}$ be a function which is one-to-one, onto, and bimeasurable (with respect to the Borel σ-fields on $\mathbf{R}^n$ and $\mathbf{R}$). Also, let $\phi_{\mathbf{N}} : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ be a function which is one-to-one, onto, and bimeasurable (with respect to the Borel σ-fields). The existence of such functions is well known.

(c) In the context of a given family $(U_i, i \in I)$ of random variables (where I is a nonempty index set), the notation $\sigma(U_i, i \in \emptyset)$ will be defined to be the totally trivial σ-field $\{\Omega, \emptyset\}$.

Lemma 2.4. *Suppose $d \geq 2$ is an integer. Suppose L is a positive integer. Suppose*

$$0 < \varepsilon \leq r \leq 1. \tag{2.5}$$

Suppose u and v are distinct elements of $\{1, 2, \ldots, d\}$.

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ with the following properties.

$$\rho'(X, 1) \leq \varepsilon \ ; \tag{2.6}$$

$$\rho_1^*(X, 1) = r \ ; \tag{2.7}$$

$$\rho(\sigma(X_{L\mathbf{e}(u)}, X_{L\mathbf{e}(v)}), \sigma(X_{\mathbf{0}})) \ = \ r \ ; \quad \text{and} \tag{2.8}$$

$$\rho_1^*(X, L+1) \ = \ 0 \ . \tag{2.9}$$

Proof. The proof will be broken into several short "steps."

Step 1. Permuting the indices if necessary, we assume without loss of generality that

$$u = 1 \quad \text{and} \quad v = 2. \tag{2.10}$$

Referring to (2.5) and applying Lemma 2.2, let $V := (V_1, V_2, V_3)$ be a random vector that satisfies (2.1), (2.2), (2.3), and (2.4). Let $W_k := (W_{k,1}, W_{k,2}, W_{k,3})$, $k \in \mathbf{Z}^d$ be a family of independent (and identically distributed) random vectors such that

$$\forall k \in \mathbf{Z}^d, \quad \mathcal{L}(W_k) = \mathcal{L}(V). \tag{2.11}$$

Refer to Definition 2.3(b). Define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad X_k := \phi_3(W_{k-L\mathbf{e}(1),1}, W_{k,2}, W_{k-L\mathbf{e}(2),3}). \tag{2.12}$$

By an elementary argument (basically just observing that X is a "moving function" of the random field $(W_k, k \in \mathbf{Z}^d)$ of i.i.d. random vectors), this random field X is strictly stationary. Also, by an elementary argument,

$$\forall k \in \mathbf{Z}^d, \quad \sigma(X_k) = \sigma(W_{k-L\mathbf{e}(1),1}, W_{k,2}, W_{k-L\mathbf{e}(2),3}). \tag{2.13}$$

Step 2. This "step" will give in detail a "template" that will be used in Steps 3–5 in the proofs of (2.6)-(2.9).

Suppose S and T are any two nonempty, disjoint subsets of $\mathbf{Z}^d$. (In Steps 3–5, special choices of such sets will be considered.)

Let Λ denote the set of all ordered pairs $(j, i) \in \mathbf{Z}^d \times \{1, 2, 3\}$ with (exactly) one of the following three properties (i)(ii)(iii):
(i) there exists $k \in S$ such that $(j, i) = (k - L\mathbf{e}(1), 1)$,
(ii) there exists $k \in S$ such that $(j, i) = (k, 2)$, or
(iii) there exists $k \in S$ such that $(j, i) = (k - L\mathbf{e}(2), 3)$.

Then by an elementary argument,

$$\sigma(X_k, k \in S) = \sigma(W_{j,i}, (j, i) \in \Lambda). \tag{2.14}$$

Trivially Λ is the set of all ordered pairs $(j, i) \in \mathbf{Z}^d \times \{1, 2, 3\}$ such that (exactly) one of the following three conditions holds:

$$j + L\mathbf{e}(1) \in S \text{ and } i = 1 \ ; \tag{2.15}$$

$$j \in S \text{ and } i = 2 \ ; \tag{2.16}$$

$$j + L\mathbf{e}(2) \in S \text{ and } i = 3 \ . \tag{2.17}$$

For each $j \in \mathbf{Z}^d$, let Λ_j denote the (possibly empty) set of all $i \in \{1, 2, 3\}$ such that $(j, i) \in \Lambda$. Then by (2.14) and an elementary argument,

$$\sigma(X_k, k \in S) \ = \bigvee_{j \in \mathbf{Z} \uparrow d} \sigma(W_{j,i}, i \in \Lambda_j) \ . \tag{2.18}$$

Here and in various places elsewhere, $\mathbf{Z}^d$ is written $\mathbf{Z} \uparrow d$ for typographical convenience. Recall from Definition 2.3(c) the convention that if the set Λ_j is empty, then $\sigma(W_{j,i}, i \in \Lambda_j) := \{\Omega, \emptyset\}$.

Now define the set $\Gamma \subset \mathbf{Z}^d \times \{1, 2, 3\}$ in relation to the set T the same way Λ was defined in relation to S. To shorten this a little, let Γ denote the set of all ordered pairs $(j, i) \in \mathbf{Z}^d \times \{1, 2, 3\}$ such that (exactly) one of the following three conditions holds:

$$j + L\mathbf{e}(1) \in T \text{ and } i = 1 \ ; \tag{2.19}$$

$$j \in T \text{ and } i = 2 \ ; \tag{2.20}$$

$$j + L\mathbf{e}(2) \in T \text{ and } i = 3 \ . \tag{2.21}$$

(These are analogous to (2.15)–(2.17).) For each $j \in \mathbf{Z}^d$, let Γ_j denote the (possibly empty) set of all $i \in \{1, 2, 3\}$ such that $(j, i) \in \Gamma$. Then analogously to (2.18),

$$\sigma(X_k, k \in T) \ = \bigvee_{j \in \mathbf{Z} \uparrow d} \sigma(W_{j,i}, i \in \Gamma_j) \ . \tag{2.22}$$

One has that

$$\forall j \in \mathbf{Z}^d, \quad \Lambda_j \cap \Gamma_j \;=\; \emptyset. \tag{2.23}$$

This is trivial. For example, if for some j one were to have $3 \in \Lambda_j \cap \Gamma_j$, then one would have $(j,3) \in \Lambda$ and $(j,3) \in \Gamma$, and hence $j + L\mathbf{e}(2) \in S$ and $j + L\mathbf{e}(2) \in T$, but that would contradict the stipulation (in the second paragraph of Step 2 here) that S and T are disjoint.

By (2.18), (2.22), and Lemma 2.1,

$$\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \;=\; \sup_{j \in \mathbf{Z} \uparrow d} \rho(\sigma(W_{j,i}, i \in \Lambda_j), \sigma(W_{j,i}, i \in \Gamma_j)) \;. \tag{2.24}$$

In Steps 3, 4, and 5 below, to complete the proof of Lemma 2.4, we shall respectively prove (2.7) and (2.8) together, then (2.9), and then (2.6).

Step 3. Proof of (2.7)–(2.8). Recall the assumption (2.10). Of course trivially

$$\rho_1^*(X, 1) \geq \rho(\sigma(X_{L\mathbf{e}(1)}, X_{L\mathbf{e}(2)}), \sigma(X_{\mathbf{0}})) \;.$$

Hence, in order to complete the proof of both (2.7) and (2.8), it suffices to prove that

$$\rho_1^*(X, 1) \leq r \quad \text{and} \tag{2.25}$$

$$\rho(\sigma(X_{L\mathbf{e}(1)}, X_{L\mathbf{e}(2)}), \sigma(X_{\mathbf{0}})) \geq r \;. \tag{2.26}$$

We shall prove (2.26) first. By (2.13), (2.11) (and its entire paragraph), and (2.3),

$$\rho(\sigma(X_{L\mathbf{e}(1)}, X_{L\mathbf{e}(2)}), \sigma(X_{\mathbf{0}})) \;\geq\; \rho(\sigma(W_{\mathbf{0},1}, W_{\mathbf{0},3}), \sigma(W_{\mathbf{0},2})) \;=\; \rho(\sigma(V_1, V_3), \sigma(V_2)) \;=\; r \;.$$

Thus (2.26) holds.

Now let us prove (2.25). Suppose S and T are any two nonempty, disjoint subsets of $\mathbf{Z}^d$. To complete the proof of (2.25), it suffices to show that

$$\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \;\leq\; r \;. \tag{2.27}$$

For this purpose, we use the framework of Step 2.

Refer again to (2.5). In the context of Lemma 2.2, one trivially has that for any two (possibly empty) disjoint sets $A, B \subset \{1, 2, 3\}$, $\rho(\sigma(V_i, i \in A), \sigma(V_i, \in B)) \leq r$. Hence by (2.11) and (2.23),

$$\forall j \in \mathbf{Z}^d, \quad \rho(\sigma(W_{j,i}, i \in \Lambda_j), \sigma(W_{j,i}, i \in \Gamma_j)) \leq r \;.$$

Hence by (2.24), eq. (2.27) holds. This completes the proof of (2.25). That in turn completes the proof of both (2.7) and (2.8).

Step 4. Proof of (2.9). Suppose S and T are nonempty subsets of $\mathbf{Z}^d$ such that

$$\mathrm{dist}_1(S,T) \geq L+1. \tag{2.28}$$

To prove (2.9), our task is to show that

$$\rho(\sigma(X_k, k \in S),\, \sigma(X_k, k \in T)) \;=\; 0\ .$$

We shall use the framework of Step 2, with the extra condition (2.28). Suppose j is any element of $\mathbf{Z}^d$. By (2.24), to complete the proof of (2.9), it suffices to show that

$$\rho(\sigma(W_{j,i}, i \in \Lambda_j),\, \sigma(W_{j,i}, i \in \Gamma_j)) \;= 0\ . \tag{2.29}$$

If the set Λ_j is empty, then $\sigma(W_{j,i}, i \in \Lambda_j) = \{\Omega, \emptyset\}$ and (2.29) holds trivially. Similarly (2.29) holds trivially if the set Γ_j is empty. Therefore, assume that neither Λ_j nor Γ_j is empty. In what follows, keep in mind (2.23).

Now consider the case where $1 \in \Lambda_j$ and $2 \in \Gamma_j$. Then (recall the terminology of Step 2) $(j,1) \in \Lambda$ and hence (see (2.15)–(2.17)) $j + L\mathbf{e}(1) \in S$. Similarly $(j,2) \in \Gamma$ and hence (see (2.19)-(2.21)) $j \in T$. Since $\|(j + L\mathbf{e}(1)) - j\|_1 = \|L\mathbf{e}(1)\|_1 = L$, one has that $\mathrm{dist}_1(S,T) \leq L$. But that contradicts (2.28). Hence it cannot be the case that $1 \in \Lambda_j$ and $2 \in \Gamma_j$.

By an analogous argument, it cannot be the case that $3 \in \Lambda_j$ and $2 \in \Gamma_j$. Hence it cannot be the case at all that $2 \in \Gamma_j$. By an analogous argument, it cannot be the case that $2 \in \Lambda_j$. The only remaining cases are when the two sets Λ_j and Γ_j are just $\{1\}$ and $\{3\}$ (in either order).

Consider the case $\Lambda_j = \{1\}$ and $\Gamma_j = \{3\}$. Then the left side of (2.29) is $\rho(\sigma(W_{j,1}), \sigma(W_{j,3}))$, which equals 0 by (2.11) (and its entire paragraph) and (2.4). Thus (2.29) holds. Similarly (2.29) holds if $\Lambda_j = \{3\}$ and $\Gamma_j = \{1\}$. Consequently, (2.29) holds in all cases. That completes the proof of (2.9).

Step 5. Proof of (2.6). Again we use the framework of Step 2, but this time with the additional assumption that (see Notation 1.1(b)) there exists $a \in \{1, 2, \dots, d\}$ and $(G, H) \in \mathcal{E}(1)$ (with such a, G, H henceforth fixed) such that

$$S = \{k \in \mathbf{Z}^d : k_a \in G\} \quad \text{and} \quad T = \{k \in \mathbf{Z}^d : k_a \in H\}\ . \tag{2.30}$$

To complete the proof of (2.6), it suffices to prove that

$$\rho(\sigma(X_k, k \in S),\, \sigma(X_k, k \in T)) \;\leq\; \varepsilon\ . \tag{2.31}$$

Without loss of generality, we assume that $G \cup H = \mathbf{Z}$ and hence $S \cup T = \mathbf{Z}^d$.

Suppose j is any element of $\mathbf{Z}^d$. By (2.24), to complete the proof of (2.31) and thereby that of (2.6), it suffices to prove that

$$\rho(\sigma(W_{j,i}, i \in \Lambda_j),\, \sigma(W_{j,i}, i \in \Gamma_j)) \;\leq\; \varepsilon\ . \tag{2.32}$$

By the sentence after (2.31), either $j \in S$ or $j \in T$. The arguments for the two cases are exactly analogous. We shall give the argument here for the case where

$$j \in S. \tag{2.33}$$

Then by (2.30),

$$j_a \in G. \tag{2.34}$$

Now let us consider three cases according to whether $a = 1$, $a = 2$, or ($d \geq 3$ and) $a \in \{3, 4, \ldots, d\}$.

Case 1: $a = 1$. Then $j_1 \in G$ by (2.34). Let $\ell := j + L\mathbf{e}(2)$. Then $\ell_1 = j_1 \in G$, hence $\ell \in S$ by (2.30), and hence $(j, 3) \in \Lambda$ by (2.17). Hence $3 \in \Lambda_j$. Also $(j, 2) \in \Lambda$ by (2.33) and (2.16); and hence $2 \in \Lambda_j$. Thus $\{2, 3\} \subset \Lambda_j$.

Recall (2.23). If $1 \in \Gamma_j$, then by (2.11) (and its entire paragraph) and (2.1),

$$\begin{aligned} \rho(\sigma(W_{j,i}, i \in \Lambda_j),\, \sigma(W_{j,i}, i \in \Gamma_j)) \;&=\; \rho(\sigma(W_{j,2}, W_{j,3}), \sigma(W_{j,1})) \\ &=\; \rho(\sigma(V_2, V_3), \sigma(V_1)) \;\leq\; \varepsilon \;, \end{aligned}$$

and hence (2.32) holds. If instead $1 \notin \Gamma_j$, then Γ_j is empty, hence $\sigma(W_{j,i}, i \in \Gamma_j)) = \{\Omega, \emptyset\}$, and hence (2.32) holds trivially (with its left side being 0). Thus (2.32) holds for Case 1.

Case 2: $a = 2$. Then $j_2 \in G$ by (2.34). Let $\ell := j + L\mathbf{e}(1)$. Then $\ell_2 = j_2 \in G$, hence $\ell \in S$ by (2.30), $(j, 1) \in \Lambda$ by (2.15), and hence $1 \in \Lambda_j$. Also $(j, 2) \in \Lambda$ by (2.33) and (2.16); and hence $2 \in \Lambda_j$. Thus $\{1, 2\} \subset \Lambda_j$.

If $3 \in \Gamma_j$, then by (2.11) and (2.2),

$$\begin{aligned} \rho(\sigma(W_{j,i}, i \in \Lambda_j),\, \sigma(W_{j,i}, i \in \Gamma_j)) \;&=\; \rho(\sigma(W_{j,1}, W_{j,2}), \sigma(W_{j,3})) \\ &=\; \rho(\sigma(V_1, V_2), \sigma(V_3)) \;\leq\; \varepsilon \;. \end{aligned}$$

and hence (2.32) holds. If instead $3 \notin \Gamma_j$, then Γ_j is empty and hence (2.32) holds trivially. Thus (2.32) holds for Case 2.

Case 3: ($d \geq 3$ and) $a \in \{3, 4, \ldots, d\}$. Of course $j_a \in G$ by (2.34). Hence $(j + L\mathbf{e}(1))_a = j_a \in G$ and $(j + L\mathbf{e}(2))_a = j_a \in G$. Hence $\{j, j + L\mathbf{e}(1), j + L\mathbf{e}(2)\} \subset S$ by (2.33) and (2.30), hence $\{(j, 1), (j, 2), (j, 3)\} \subset \Lambda$ by (2.15)–(2.17), and hence $\Lambda_j = \{1, 2, 3\}$. Hence Γ_j is empty by (2.23), and hence the left side of (2.32) equals 0. Thus (2.32) holds for Case 3.

That completes the argument for (2.32), for (2.31), and hence for (2.6). That completes the proof of Lemma 2.4.

Lemma 2.5. *Suppose $d \geq 2$ is an integer. Suppose L is a positive integer. Suppose (2.5) holds.*

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) eqs. (2.6), (2.7), and (2.9) hold and (ii) for every ordered pair $(u, v) \in \{1, 2, \ldots, d\}^2$ such that $u < v$, eq. (2.8) holds.

Proof. For each ordered pair $(u, v) \in \{1, 2, \ldots, d\}^2$ such that $u < v$, applying Lemma 2.4, let $X^{(u,v)} := (X_k^{(u,v)}, k \in \mathbf{Z}^d)$ be a strictly stationary random field such that (2.6)–(2.9) hold with $X^{(u,v)}$ in place of X. Let these random fields $X^{(u,v)}$ be constructed together in such a way that they are independent of each other. Referring to Definition 2.3(b), define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows: For each $k \in \mathbf{Z}^d$,

$$\begin{aligned} X_k \; := \; \phi_{d(d-1)/2}\Big(& X_k^{(1,2)}, X_k^{(1,3)}, \ldots, X_k^{(1,d)};\; X_k^{(2,3)}, X_k^{(2,4)}, \ldots, X_k^{(2,d)}; \\ & X_k^{(3,4)}, X_k^{(3,5)}, \ldots, X_k^{(3,d)};\; \ldots;\; X_k^{(d-2,d-1)}, X_k^{(d-2,d)};\; X_k^{(d-1,d)} \Big) \, . \end{aligned} \tag{2.35}$$

By an elementary argument, the random field X is strictly stationary. Also, by an elementary argument, for any nonempty set $S \subset \mathbf{Z}^d$,

$$\sigma(X_k, k \in S) = \bigvee \sigma(X_k^{(u,v)}, k \in S) \tag{2.36}$$

where the join is taken over all ordered pairs $(u, v) \in \{1, 2, \ldots, d\}^2$ such that $u < v$. Hence by Lemma 2.1, for any two nonempty, disjoint sets $S, T \subset \mathbf{Z}^d$,

$$\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \;=\; \sup\; \rho(\sigma(X_k^{(u,v)}, k \in S), \sigma(X_k^{(u,v)}, k \in T)) \tag{2.37}$$

where the supremum is taken over all such ordered pairs (u, v).

By (2.37) and the entire paragraph containing (2.35), the random field X satisfies (2.6), (2.7), and (2.9). Also, for any given ordered pair $(u, v) \in \{1, 2, \ldots, d\}^2$ such that $u < v$, one now has by (2.8) for $X^{(u,v)}$, (2.36), and (2.7) for X itself (see the preceding sentence) that

$$\begin{aligned} r \;&=\; \rho(\sigma(X_{L\mathbf{e}(u)}^{(u,v)}, X_{L\mathbf{e}(v)}^{(u,v)}), \sigma(X_{\mathbf{0}}^{(u,v)})) \\ &\le\; \rho(\sigma(X_{L\mathbf{e}(u)}, X_{L\mathbf{e}(v)}), \sigma(X_{\mathbf{0}})) \;\le\; \rho_1^*(X, 1) \;=\; r \, , \end{aligned} \tag{2.38}$$

and thus equality must hold in (2.38), and hence the random field X itself satisfies (2.8). That completes the proof of Lemma 2.5.

3. Part 2 of proof of Theorem 1.9

The goal of this section is to provide, in Lemma 3.2 below, another key "building block" random field for the final construction in Section 5. Lemma 3.2 will be achieved through an intermediate step, in Lemma 3.1. The lemmas here and in Section 4 still hold for, and will include, the case $d = 1$. They will be used in Section 5 only for $d \ge 2$.

Lemma 3.1. *Suppose d is a positive integer. Suppose L is a positive integer. Suppose*

$$0 < \varepsilon \le r \le 1. \tag{3.1}$$

Suppose u is an element of $\{1, 2, \ldots, d\}$.

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ with the following properties.

$$\rho(X, 1) \le \varepsilon \ ; \tag{3.2}$$

$$\rho_1^*(X, 1) = r \ ; \tag{3.3}$$

$$\rho(\sigma(X_{L\mathbf{e}(u)}, X_{-L\mathbf{e}(u)}), \sigma(X_{\mathbf{0}})) \ = \ r \ ; \quad \text{and} \tag{3.4}$$

$$\rho_1^*(X, L+1) \ = \ 0 \ . \tag{3.5}$$

Proof. The proof is similar to that of Lemma 2.4, and it will be "structured" in a similar manner. As we go along, we shall omit some of the details.

Step 1. Permuting the indices if necessary, we assume without loss of generality that

$$u = 1. \tag{3.6}$$

Referring to (3.1) and applying Lemma 2.2, let $V := (V_1, V_2, V_3)$ be a random vector that satisfies (2.1), (2.2), (2.3), and (2.4). Let $W_k := (W_{k,1}, W_{k,2}, W_{k,3})$, $k \in \mathbf{Z}^d$ be a family of independent (and identically distributed) random vectors such that (as in (2.11))

$$\forall k \in \mathbf{Z}^d, \quad \mathcal{L}(W_k) = \mathcal{L}(V). \tag{3.7}$$

Refer to Definition 2.3(b). Define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad X_k := \phi_3(W_{k-L\mathbf{e}(1),1}, W_{k,2}, W_{k+L\mathbf{e}(1),3}). \tag{3.8}$$

This random field X is strictly stationary, and

$$\forall k \in \mathbf{Z}^d, \quad \sigma(X_k) = \sigma(W_{k-L\mathbf{e}(1),1}, W_{k,2}, W_{k+L\mathbf{e}(1),3}). \tag{3.9}$$

Step 2. This "step" is very nearly identical to Step 2 in the proof of Lemma 2.4. Again we suppose S and T are any two nonempty, disjoint subsets of $\mathbf{Z}^d$. In the rest of Step 2 in the proof of Lemma 2.4, we make only the following changes:

In the third paragraph of Step 2, property (iii) is changed to the following:
(iii′) there exists $k \in S$ such that $(j, i) = (k + L\mathbf{e}(1), 3)$.

Eq. (2.17) is replaced by

$$j - L\mathbf{e}(1) \in S \text{ and } i = 3. \tag{2.17′}$$

Eq. (2.21) is replaced by

$$j - L\mathbf{e}(1) \in T \text{ and } i = 3. \tag{2.21′}$$

Other than these changes, the definitions and the numbered displayed equations (2.14)–(2.24) in Step 2 of the proof of Lemma 2.4 hold verbatim here (with essentially the same arguments). In the rest of the proof of Lemma 3.1, we shall refer freely to the numbered displayed equations (2.14)–(2.24), except of course that (2.17′) and (2.21′) above replace (2.17) and (2.21).

In the next three steps, we shall prove (3.3)-(3.4), then (3.5), and then (3.2).

Step 3. Proof of (3.3)–(3.4). Recall the assumption (3.6). Of course trivially

$$\rho_1^*(X,1) \ge \rho(\sigma(X_{L\mathbf{e}(1)}, X_{-L\mathbf{e}(1)}),\, \sigma(X_{\mathbf{0}})) \ .$$

Hence, in order to complete the proof of both (3.3) and (3.4), it suffices to prove that

$$\rho_1^*(X,1) \le r \quad \text{and} \tag{3.10}$$

$$\rho(\sigma(X_{L\mathbf{e}(1)}, X_{-L\mathbf{e}(1)}),\, \sigma(X_{\mathbf{0}})) \ge r \ . \tag{3.11}$$

The proofs of these two equations are essentially identical to those of eqs. (2.25) and (2.26) in Step 3 of the proof of Lemma 2.4. First, in adapting the proof of (2.26) to verify (3.11), (i) in the displayed equation in the second line after (2.26), one just changes the subscript $L\mathbf{e}(2)$ to $-L\mathbf{e}(1)$, and (ii) in the line right after (2.26), instead of citing (2.13) one cites (3.9). The proof of (3.10) is essentially identical to that of (2.25) (recall that (3.7) is identical to (2.11)). Thus (3.10) and (3.11) hold, and hence (3.3) and (3.4) hold.

Step 4. Proof of (3.5). This is essentially identical to the proof of (2.9) in Step 4 in the proof of Lemma 2.4. (Recall that (2.17′) and (2.21′) above are used in place of (2.17) and (2.21).) The argument need not be repeated here.

Step 5. Proof of (3.2). This will follow the argument in Step 5 (proof of (2.6)) in the proof of Lemma 2.4, but with a few minor differences:

In place of the first sentence (including (eq. (2.30)) in Step 5 in the proof of Lemma 2.4, we assume instead that there exists $a \in \{1, 2, \dots, d\}$ such that

$$S = \{k \in \mathbf{Z}^d : k_a \le 0\} \quad \text{and} \quad T = \{k \in \mathbf{Z}^d : k_a \ge 1\} \ . \tag{2.30′}$$

The line after (2.31) will now read simply $S \cup T = \mathbf{Z}^d$. Eqs. (2.31), (2.32), and (2.33) remain unchanged, and they play the same roles as before: Our goal is to prove (2.31); and to accomplish that, our goal is to prove (2.32), for an arbitrary fixed $j \in \mathbf{Z}^d$. Eq. (2.33) ($j \in S$) is an assumption made without essential loss of generality. (The argument for (2.32) for the case $j \in T$ is analogous.)

Eq. (2.34) is (in accordance with (2.30′) above and (2.33)) changed to

$$j_a \le 0. \tag{2.34′}$$

Instead of the three cases mentioned after (2.34), we consider just two cases according to whether $a = 1$ or ($d \ge 2$ and) $a \in \{2, 3, \dots, d\}$.

Case 1: $a = 1$. Then $j_1 \le 0$ by (2.34′). Let $\ell := j - L\mathbf{e}(1)$. Then $\ell_1 = j_1 - L < 0$ and hence $\ell \in S$ by (2.30′), and hence $(j, 3) \in \Lambda$ by (2.17′) above. (Recall again the paragraph containing (2.15)–(2.17) in Section 2; and again recall that in our context here, (2.17′) replaces (2.17).) Hence $3 \in \Lambda_j$. Also $(j, 2) \in \Lambda$ by (2.33) and (2.16) (which was not changed); and hence $2 \in \Lambda_j$. Thus $\{2, 3\} \subset \Lambda_j$.

The rest of the argument for showing (2.32) for Case 1 in Step 5 in the proof of Lemma 2.4 carries over essentially verbatim here. (Keep in mind that (3.7) is identical to (2.11).) Thus (2.32) holds for Case 1 here.

Case 2: ($d \geq 2$ and) $a \in \{2, 3, \dots, d\}$. This is similar to Case 3 in Step 5 in the proof of Lemma 2.4, but we shall give the details: First, $j_a \leq 0$ by (2.34′). Hence $(j + L\mathbf{e}(1))_a = j_a \leq 0$ and $(j - L\mathbf{e}(1))_a = j_a \leq 0$. Hence $\{j, j + L\mathbf{e}(1), j - L\mathbf{e}(1)\} \subset S$ by (2.33) and (2.30′). Hence $\{(j, 1), (j, 2), (j, 3)\} \subset \Lambda$ by (2.15), (2.16), and (2.17′); and hence $\Lambda_j = \{1, 2, 3\}$. Hence Γ_j is empty by (2.23), and hence the left side of (2.32) is 0. Thus (2.32) holds for Case 2 here.

That completes the argument for (2.32), (2.31), and thereby for (3.2). That completes the proof of Lemma 3.1.

Lemma 3.2. *Suppose d is a positive integer. Suppose L is a positive integer. Suppose (3.1) holds.*

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) eqs. (3.2), (3.3), and (3.5) hold and (ii) for every element $u \in \{1, 2, \dots, d\}$, eq. (3.4) holds.

Proof. This is similar to the proof of Lemma 2.5, but there are a few slight differences, and therefore we shall give the argument.

For each element $u \in \{1, 2, \dots, d\}$, applying Lemma 3.1, let $X^{(u)} := (X_k^{(u)}, k \in \mathbf{Z}^d)$ be a strictly stationary random field such that (3.2)–(3.5) hold with $X^{(u)}$ in place of X. Let these random fields $X^{(u)}$ be constructed together in such a way that they are mutually independent. Referring to Definition 2.3(b), define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows: For each $k \in \mathbf{Z}^d$,

$$X_k \ := \ \phi_d\Big(X_k^{(1)}, X_k^{(2)}, \dots, X_k^{(d)}\Big). \tag{3.12}$$

By an elementary argument, the random field X is strictly stationary. Also, by an elementary argument, for any nonempty subset $S \subset \mathbf{Z}^d$,

$$\sigma(X_k, k \in S) = \bigvee_{j \in \{1,2,\dots,d\}} \sigma(X_k^{(u)}, k \in S). \tag{3.13}$$

Hence by Lemma 2.1, for any two nonempty, disjoint sets $S, T \subset \mathbf{Z}^d$,

$$\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \ = \sup_{j \in \{1,2,\dots,d\}} \rho(\sigma(X_k^{(u)}, k \in S), \sigma(X_k^{(u)}, k \in T)). \tag{3.14}$$

By (3.14) and the entire paragraph containing (3.12), the random field X satisfies (3.2), (3.3), and (3.5). Also, for any given $u \in \{1, 2, \dots, d\}$, one now has by (3.4) for $X^{(u)}$, (3.13), and (3.3) for X itself (see the preceding sentence) that

$$\begin{aligned} r &= \rho(\sigma(X^{(u)}_{\mathbf{Le}(u)}, X^{(u)}_{-\mathbf{Le}(u)}),\, \sigma(X^{(u)}_{\mathbf{0}})) \\ &\le \rho(\sigma(X_{\mathbf{Le}(u)}, X_{-\mathbf{Le}(u)}),\, \sigma(X_{\mathbf{0}})) \;\le\; \rho_1^*(X,1) \;=\; r\ , \end{aligned} \tag{3.15}$$

and thus equality must hold in (3.15), and hence the random field X itself satisfies (3.4). That completes the proof of Lemma 3.2.

4. Part 3 of proof of Theorem 1.9

The goal of this section is to provide, in Lemma 4.2 below, another key "building block" random field for the final construction in Section 5. Lemma 4.2 will be achieved through an intermediate step, in Lemma 4.1.

Lemma 4.1. *Suppose d is a positive integer. Suppose L is a positive integer. Suppose*

$$0 < r \le 1. \tag{4.1}$$

Suppose u is an element of $\{1, 2, \dots, d\}$.

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ with the following properties.

$$\rho_1^*(X,1) = r\ ; \tag{4.2}$$

$$\rho(\sigma(X_{\mathbf{Le}(u)}),\, \sigma(X_{\mathbf{0}})) \;=\; 4\alpha(\sigma(X_{\mathbf{Le}(u)}),\, \sigma(X_{\mathbf{0}})) \;=\; r\ ; \quad \text{and} \tag{4.3}$$

$$\rho_1^*(X, L+1) \;=\; 0\ . \tag{4.4}$$

At least the spirit of this lemmas has been shown as part of a more complicated construction in [5]. For simplicity, we shall give a self-contained proof of Lemma 4.1 here.

Proof. Again the proof will be broken into several small "steps."

Step 1. Permuting the indices if necessary, we assume without loss of generality that

$$u = 1. \tag{4.5}$$

Let $V := (V_1, V_2)$ be a random vector whose distribution is as follows:

$$\begin{aligned} &P(V_1 = V_2 = 1) \;=\; P(V_1 = V_2 = 0) \;=\; (1/4)(1+r) \quad \text{and} \\ &P(V_1 = 1, V_2 = 0) \;=\; P(V_1 = 0, V_2 = 1) \;=\; (1/4)(1-r)\ . \end{aligned} \tag{4.6}$$

By a simple calculation, $\mathrm{Corr}(V_1, V_2) = r$. By [7, V1, Proposition 3.20],

$$\rho(\sigma(V_1), \sigma(V_2)) \;=\; 4\alpha(\sigma(V_1), \sigma(V_2)) \;=\; r. \tag{4.7}$$

Let $W_k := (W_{k,1}, W_{k,2})$, $k \in \mathbf{Z}^d$ be a family of independent (and identically distributed) random vectors that take their values in $\{0,1\}^2$, such that

$$\forall k \in \mathbf{Z}^d, \quad \mathcal{L}(W_k) = \mathcal{L}(V). \tag{4.8}$$

Define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad X_k := 2W_{k,1} + W_{k-L\mathbf{e}(1),2}\ . \tag{4.9}$$

This random field X is strictly stationary. Also, by an elementary argument,

$$\forall k \in \mathbf{Z}^d, \quad \sigma(X_k) = \sigma(W_{k,1}, W_{k-L\mathbf{e}(1),2}). \tag{4.10}$$

(For example, $\{W_{k-L\mathbf{e}(1),2} = 1\} = \{X_k = 1 \text{ or } 3\}$.)

Step 2. This will be analogous to an abridged version of Step 2 in the proof of Lemma 2.4. Because of several difference in our context here, we shall spell out the main details.

Suppose S and T are any two nonempty, disjoint subsets of $\mathbf{Z}^d$.

Let Λ denote the set of all ordered pairs $(j,i) \in \mathbf{Z}^d \times \{1,2\}$ such that (exactly) one of the following two conditions holds:

$$j \in S \text{ and } i = 1\ ; \tag{4.11}$$

$$j + L\mathbf{e}(1) \in S \text{ and } i = 2\ . \tag{4.12}$$

For each $j \in \mathbf{Z}^d$, let Λ_j denote the (possibly empty) set of all $i \in \{1,2\}$ such that $(j,i) \in \Lambda$. Then by (4.10) and an elementary argument,

$$\sigma(X_k, k \in S) \ = \ \bigvee_{j \in \mathbf{Z}\uparrow d} \sigma(W_{j,i}, i \in \Lambda_j)\ . \tag{4.13}$$

Let Γ denote the set of all ordered pairs $(j,i) \in \mathbf{Z}^d \times \{1,2\}$ such that (exactly) one of the following two conditions holds:

$$j \in T \text{ and } i = 1\ ; \tag{4.14}$$

$$j + L\mathbf{e}(1) \in T \text{ and } i = 2\ . \tag{4.15}$$

For each $j \in \mathbf{Z}^d$, let Γ_j denote the (possibly empty) set of all $i \in \{1,2\}$ such that $(j,i) \in \Gamma$. Then analogously to (4.13),

$$\sigma(X_k, k \in T) \ = \ \bigvee_{j \in \mathbf{Z}\uparrow d} \sigma(W_{j,i}, i \in \Gamma_j)\ . \tag{4.16}$$

One has that (analogously to (2.23) in the proof of Lemma 2.4)

$$\forall j \in \mathbf{Z}^d, \quad \Lambda_j \cap \Gamma_j \ = \ \emptyset. \tag{4.17}$$

By (4.13), (4.16), and Lemma 2.1,

$$\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \ = \ \sup_{j \in \mathbf{Z} \uparrow d} \rho(\sigma(W_{j,i}, i \in \Lambda_j), \sigma(W_{j,i}, i \in \Gamma_j)) \ . \tag{4.18}$$

Now we are ready to prove (4.2)–(4.3) and then (4.4)

Step 3. Proof of (4.2)–(4.3). Recall the assumption (4.5). Of course by (1.8),

$$\rho_1^*(X, 1) \ \geq \ \rho(\sigma(X_{L\mathbf{e}(1)}), \sigma(X_{\mathbf{0}})) \ \geq \ 4\alpha(\sigma(X_{L\mathbf{e}(1)}), \sigma(X_{\mathbf{0}})) \ .$$

Hence, in order to complete the proof of both (4.2) and (4.3), it suffices to prove that

$$\rho_1^*(X, 1) \leq r \quad \text{and} \tag{4.19}$$

$$4\alpha(\sigma(X_{L\mathbf{e}(1)}), \sigma(X_{\mathbf{0}})) \geq r \ . \tag{4.20}$$

Now (4.20) holds because by (4.10), (4.8), and (4.7),

$$4\alpha(\sigma(X_{L\mathbf{e}(1)}), \sigma(X_{\mathbf{0}})) \ \geq \ 4\alpha(\sigma(W_{\mathbf{0},2}), \sigma(W_{\mathbf{0},1})) \ = \ 4\alpha(\sigma(V_2), \sigma(V_1)) \ = \ r \ .$$

Now to prove (4.19), suppose S and T are any two nonempty disjoint subsets of $\mathbf{Z}^d$. It suffices to show that $\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \leq r$.

Suppose j is any element of $\mathbf{Z}^d$. By (4.18), to complete the proof of (4.19) it suffices to show that $\rho(\sigma(W_{j,i}, i \in \Lambda_j), \sigma(W_{j,i}, i \in \Gamma_j)) \leq r$. Because of "symmetry," (4.17), and Definition 2.3(c), it suffices to prove that $\rho(\sigma(W_{j,1}), \sigma(W_{j,2})) \leq r$. But that holds (with equality) by (4.8) and (4.7). That completes the proof of (4.19). Eqs. (4.2)–(4.3) have now been proved.

Step 4. Proof of (4.4). Suppose S and T are nonempty subsets of $\mathbf{Z}^d$ such that

$$\text{dist}_1(S, T) \geq L + 1. \tag{4.21}$$

To prove (4.4), our task is to show that $\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) = 0$.

We shall use the framework of Step 2, with the extra condition (4.21). Suppose j is any element of $\mathbf{Z}^d$. By (4.18), to complete the proof of (4.4), it suffices to show that

$$\rho(\sigma(W_{j,i}, i \in \Lambda_j), \sigma(W_{j,i}, i \in \Gamma_j)) \ = 0 \ .$$

By definition 2.3(c), it suffices to show that at least one of the two sets Λ_j and Γ_j is empty.

Refer to (4.17). If (say) $\Lambda_j = \{1\}$ and $\Gamma_j = \{2\}$, then one would have $(j, 1) \in \Lambda$, $(j, 2) \in \Gamma$, and hence by (4.11) and (4.15) one would have $j \in S$ and $j + L\mathbf{e}(1) \in T$, but that contradicts (4.21). An analogous contradiction arises if $\Lambda_j = \{2\}$ and $\Gamma_j = \{1\}$. Hence at least one of the sets Λ_j and Γ_j is empty. That completes the proof of (4.4). That completes the proof of Lemma 4.1.

Lemma 4.2. *Suppose d is a positive integer. Suppose L is a positive integer. Suppose (4.1) holds.*

Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) eqs. (4.2) and (4.4) hold and (ii) for every element $u \in \{1, 2, \dots, d\}$, eq. (4.3) holds.

Lemma 4.2 is derived from Lemma 4.1 in essentially the same way that Lemma 3.2 was derived from Lemma 3.1. The argument need not be given here.

5. Part 4 of proof of Theorem 1.9

In this section, Theorem 1.9 will be proved. The assumptions in that theorem will be spelled out again here for convenient reference:

$$d \in \{2, 3, 4, \dots\} \ ; \tag{5.1}$$
$$a_1 \ge a_2 \ge a_3 \ge \dots \ ; \tag{5.2}$$
$$b_1 \ge b_2 \ge b_3 \ge \dots \ ; \tag{5.3}$$
$$c_1 \ge c_2 \ge c_3 \ge \dots \ ; \tag{5.4}$$
$$\forall n \in \mathbf{N}, \quad 0 < a_n \le b_n \le c_n \le 1 \ . \tag{5.5}$$

Three more lemmas will be needed, building on Lemmas 2.5, 3.2, and 4.2 respectively.

Lemma 5.1. *Under the assumptions (5.1)–(5.5), there exists a strictly stationary random field $Y := (Y_k, k \in \mathbf{Z}^d)$ with the following properties:*

$$\forall n \in \mathbf{N}, \quad \rho(Y, n) \le \rho'(Y, n) \le a_n \ ; \tag{5.6}$$
$$\forall n \in \mathbf{N}, \quad \rho_1^*(Y, n) = c_n \ ; \tag{5.7}$$

and for every positive integer n and every ordered pair $(u, v) \in \{1, 2, \dots, d\}^2$ such that $u < v$,

$$\rho(\sigma(Y_{n\mathbf{e}(u)}, Y_{n\mathbf{e}(v)}), \, \sigma(Y_{\mathbf{0}})) \ = \ c_n \ . \tag{5.8}$$

Proof. For each positive integer m, applying (5.1), (5.5), and Lemma 2.5, let $Y^{(m)} := (Y_k^{(m)}, k \in \mathbf{Z}^d)$ be a strictly stationary random field such that the following conditions hold:

$$\rho'(Y^{(m)}, 1) \le a_m; \tag{5.9}$$
$$\rho_1^*(Y^{(m)}, 1) = c_m; \tag{5.10}$$
$$\rho_1^*(Y^{(m)}, m + 1) = 0; \tag{5.11}$$

and for every ordered pair $(u, v) \in \{1, 2, \dots, d\}^2$ such that $u < v$,

$$\rho(\sigma(Y^{(m)}_{m\mathbf{e}(u)}, Y^{(m)}_{m\mathbf{e}(v)}), \, \sigma(Y^{(m)}_{\mathbf{0}})) = c_m. \tag{5.12}$$

Let these random fields $Y^{(m)}$, $m \in \mathbf{N}$ be constructed in such as way that they are independent of each other.

Referring to Definition 2.3(b), define the random field $Y := (Y_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad Y_k := \phi_{\mathbf{N}}(Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}, \dots) \ . \tag{5.13}$$

By an elementary argument, this random field Y is strictly stationary. Also, by an elementary argument,

$$\forall k \in \mathbf{Z}^d, \quad \sigma(Y_k) = \sigma(Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}, \dots) \ . \tag{5.14}$$

By Lemma 2.1, for any two nonempty, disjoint sets $S, T \subset \mathbf{Z}^d$,

$$\rho(\sigma(Y_k, k \in S), \sigma(Y_k, k \in T)) \ = \ \sup_{m \in \mathbf{N}} \rho(\sigma(Y_k^{(m)}, k \in S), \sigma(Y_k^{(m)}, k \in T)) \ . \tag{5.15}$$

Hence by a simple argument, for every positive integer n,

$$\rho(Y, n) \ = \ \sup_{m \in \mathbf{N}} \rho(Y^{(m)}, n) \ ; \tag{5.16}$$

$$\rho'(Y, n) \ = \ \sup_{m \in \mathbf{N}} \rho'(Y^{(m)}, n) \ ; \quad \text{and} \tag{5.17}$$

$$\rho_1^*(Y, n) \ = \ \sup_{m \in \mathbf{N}} \rho_1^*(Y^{(m)}, n) \ . \tag{5.18}$$

Proof of (5.6). Suppose $n \in \mathbf{N}$. For each $m \geq n$, by (5.9) and (5.2),

$$\rho'(Y^{(m)}, n) \ \leq \ \rho'(Y^{(m)}, 1) \ \leq \ a_m \ \leq \ a_n \ . \tag{5.19}$$

If $n \geq 2$, then for each $m \leq n-1$, by ((1.15) and (1.14) and) (5.11),

$$\rho'(Y^{(m)}, n) \ \leq \ \rho_1^*(Y^{(m)}, n) \ \leq \ \rho_1^*(Y^{(m)}, m+1) \ = \ 0 \tag{5.20}$$

(and hence $\rho'(Y^{(m)}, n) = 0$). Hence (whether $n = 1$ or $n \geq 2$) by (5.17) (and (5.5)), $\rho'(Y, n) \leq a_n$. Thus (recall (1.15)) eq. (5.6) holds.

Proof of (5.7)-(5.8). Suppose $n \in \mathbf{N}$. For each $m \geq n$, by (5.10) and (5.4),

$$\rho_1^*(Y^{(m)}, n) \ \leq \ \rho_1^*(Y^{(m)}, 1) \ = \ c_m \ \leq \ c_n \ . \tag{5.21}$$

If $n \geq 2$, then for each $m \leq n-1$, by (5.11),

$$\rho_1^*(Y^{(m)}, n) \ \leq \ \rho_1^*(Y^{(m)}, m+1) \ = \ 0 \tag{5.22}$$

(and hence $\rho_1^*(Y^{(m)}, n) = 0$). Hence (whether $n = 1$ or $n \geq 2$) by (5.18) (and (5.5)),

$$\rho_1^*(Y, n) \leq c_n \ . \tag{5.23}$$

Now by (5.12) with $m = n$ itself, followed by (5.14) and (5.23), for any given ordered pair $(u, v) \in \{1, 2, \dots, d\}^2$ such that $u < v$ (recall (5.1)),

$$\begin{aligned} c_n &= \rho(\sigma(Y^{(n)}_{n\mathbf{e}(u)}, Y^{(n)}_{n\mathbf{e}(v)}), \sigma(Y^{(n)}_{\mathbf{0}})) \\ &\le \rho(\sigma(Y_{n\mathbf{e}(u)}, Y_{n\mathbf{e}(v)}), \sigma(Y_{\mathbf{0}})) \ \le \ \rho_1^*(Y, n) \ \le \ c_n \ , \end{aligned} \tag{5.24}$$

and hence equality must hold everywhere in (5.24). That yields (5.7) and (for every ordered pair $(u, v) \in \{1, 2, \dots, d\}^2$ such that $u < v$) (5.8) as well. That completes the proof of Lemma 5.1.

Lemma 5.2. *Under the assumptions (5.1)–(5.5), there exists a strictly stationary random field $Z := (Z_k, k \in \mathbf{Z}^d)$ with the following properties:*

$$\forall n \in \mathbf{N}, \quad \rho(Z, n) \le a_n \ ; \tag{5.25}$$

$$\forall n \in \mathbf{N}, \quad \rho'(Z, n) = \rho_1^*(Z, n) = b_n \ ; \tag{5.26}$$

and for every positive integer n and every element $u \in \{1, 2, \dots, d\}$,

$$\rho(\sigma(Z_{n\mathbf{e}(u)}, Z_{-n\mathbf{e}(u)}), \sigma(Z_{\mathbf{0}})) \ = \ b_n \ . \tag{5.27}$$

Proof. For each positive integer m, applying (5.1), (5.5), and Lemma 3.2, let $Z^{(m)} := (Z^{(m)}_k, k \in \mathbf{Z}^d)$ be a strictly stationary random field such that the following hold:

$$\rho(Z^{(m)}, 1) \le a_m; \tag{5.28}$$

$$\rho_1^*(Z^{(m)}, 1) = b_m; \tag{5.29}$$

$$\rho_1^*(Z^{(m)}, m+1) = 0; \tag{5.30}$$

and for every element $u \in \{1, 2, \dots, d\}$,

$$\rho(\sigma(Z^{(m)}_{m\mathbf{e}(u)}, Z^{(m)}_{-m\mathbf{e}(u)}), \sigma(Z^{(m)}_{\mathbf{0}})) = b_m. \tag{5.31}$$

Let these random fields $Y^{(m)}$, $m \in \mathbf{N}$ be constructed in such as way that they are independent of each other.

Referring to Definition 2.3(b), define the random field $Z := (Z_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad Z_k := \phi_{\mathbf{N}}(Z^{(1)}_k, Z^{(2)}_k, Z^{(3)}_k, \dots) \ . \tag{5.32}$$

By an elementary argument, this random field Z is strictly stationary. Also, for this random field Z, exact analogs of eqs. (5.14)–(5.18) hold. (There is no need to write them out explicitly here.) The rest of the argument will involve almost exact analogs of (5.19)–(5.24); but because of (minor but) pervasive differences, it will be given in detail.

Proof of (5.25). Suppose $n \in \mathbf{N}$. For each $m \geq n$, by (5.28) and (5.2), $\rho(Z^{(m)}, n) \leq \rho(Z^{(m)}, 1) \leq a_m \leq a_n$. If $n \geq 2$, then for each $m \leq n-1$, by ((1.15) and (1.14) and) (5.30), $\rho(Z^{(m)}, n) \leq \rho_1^*(Z^{(m)}, n) \leq \rho_1^*(Z^{(m)}, m+1) = 0$ (and hence $\rho(Z^{(m)}, n) = 0$). Hence (whether $n = 1$ or $n \geq 2$) by the analog of (5.16), $\rho(Z, n) \leq a_n$. Thus (5.25) holds.

Proof of (5.26)-(5.27). Suppose $n \in \mathbf{N}$. For each $m \geq n$, by (5.29) and (5.3), $\rho_1^*(Z^{(m)}, n) \leq \rho_1^*(Z^{(m)}, 1) = b_m \leq b_n$. If $n \geq 2$, then for each $m \leq n-1$, by (5.30), $\rho_1^*(Z^{(m)}, n) \leq \rho_1^*(Z^{(m)}, m+1) = 0$ (and hence $\rho_1^*(Z^{(m)}, n) = 0$). Hence (whether $n = 1$ or $n \geq 2$) by the analog of (5.18),

$$\rho_1^*(Z, n) \leq b_n \ . \tag{5.33}$$

Now by (5.31) with $m = n$ itself, followed by (1.14)–(1.15) and (5.33), for any element $u \in \{1, 2, \dots, d\}$,

$$\begin{aligned} b_n \ &= \ \rho(\sigma(Z^{(n)}_{n\mathbf{e}(u)}, Z^{(n)}_{-n\mathbf{e}(u)}), \, \sigma(Z^{(n)}_{\mathbf{0}})) \\ &\leq \ \rho(\sigma(Z_{n\mathbf{e}(u)}, Z_{-n\mathbf{e}(u)}), \, \sigma(Z_{\mathbf{0}})) \ \leq \ \rho'(Z, n) \ \leq \ \rho_1^*(Z, n) \ \leq \ b_n \ , \end{aligned} \tag{5.34}$$

and hence equality must hold everywhere in (5.34). That yields (5.26) and (for every element $u \in \{1, 2, \dots, d\}$) (5.27) as well. That completes the proof of Lemma 5.2.

Lemma 5.3. *Under the assumptions (5.1)–(5.5), there exists a strictly stationary random field $U := (U_k, k \in \mathbf{Z}^d)$ with the following properties:*

$$\forall n \in \mathbf{N}, \quad \rho(U, n) = \rho'(U, n) = \rho_1^*(U, n) = a_n \ ; \tag{5.35}$$

and for every positive integer n and every element $u \in \{1, 2, \dots, d\}$,

$$\rho(\sigma(U_{n\mathbf{e}(u)}), \, \sigma(U_{\mathbf{0}})) \ = \ 4\alpha(\sigma(U_{n\mathbf{e}(u)}), \, \sigma(U_{\mathbf{0}})) \ = \ a_n \ . \tag{5.36}$$

Proof. For each positive integer m, applying (5.1), (5.5), and Lemma 4.2, let $U^{(m)} := (U_k^{(m)}, k \in \mathbf{Z}^d)$ be a strictly stationary random field such that the following hold:

$$\rho_1^*(U^{(m)}, 1) = a_m; \tag{5.37}$$

$$\rho_1^*(U^{(m)}, m+1) = 0; \tag{5.38}$$

and for every element $u \in \{1, 2, \dots, d\}$,

$$\rho(\sigma(U^{(m)}_{m\mathbf{e}(u)}), \, \sigma(U^{(m)}_{\mathbf{0}})) \ = \ 4\alpha(\sigma(U^{(m)}_{m\mathbf{e}(u)}), \, \sigma(U^{(m)}_{\mathbf{0}})) \ = \ a_m \ . \tag{5.39}$$

Let these random fields $U^{(m)}$, $m \in \mathbf{N}$ be constructed in such as way that they are independent of each other.

Referring to Definition 2.3(b), define the random field $U := (U_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad U_k := \phi_{\mathbf{N}}(U_k^{(1)}, U_k^{(2)}, U_k^{(3)}, \dots) \ . \tag{5.40}$$

By an elementary argument, this random field U is strictly stationary. Also, for this random field U, exact analogs of eqs. (5.14)–(5.18) hold. The rest of the argument will involve almost exact analogs of (5.21)–(5.24); but again the details will be given.

Proof of (5.35)-(5.36). Suppose $n \in \mathbf{N}$. For each $m \geq n$, by (5.37) and (5.2), $\rho_1^*(U^{(m)}, n) \leq \rho_1^*(U^{(m)}, 1) = a_m \leq a_n$. If $n \geq 2$, then for each $m \leq n-1$, by (5.38), $\rho_1^*(U^{(m)}, n) \leq \rho_1^*(U^{(m)}, m+1) = 0$ (and hence $\rho_1^*(U^{(m)}, n) = 0$). Hence (whether $n = 1$ or $n \geq 2$) by the analog of (5.18),

$$\rho_1^*(U, n) \leq a_n \ . \tag{5.41}$$

Now by (5.39) with $m = n$ itself, followed by (1.14)–(1.15) and (5.41), for any given element $u \in \{1, 2, \dots, d\}$,

$$\begin{aligned} a_n \ &= \ 4\alpha(\sigma(U^{(n)}_{\mathbf{ne}(u)}), \sigma(U^{(n)}_{\mathbf{0}})) \ = \ \rho(\sigma(U^{(n)}_{\mathbf{ne}(u)}), \sigma(U^{(n)}_{\mathbf{0}})) \\ &\leq \ \rho(\sigma(U_{\mathbf{ne}(u)}), \sigma(U_{\mathbf{0}})) \ \leq \ \rho(U, n) \ \leq \ \rho'(U, n) \ \leq \ \rho_1^*(U, n) \ \leq \ a_n \ , \end{aligned} \tag{5.42}$$

and hence equality must hold everywhere in (5.42). That yields (5.35) and (for every element $u \in \{1, 2, \dots, d\}$) (5.36) as well. That completes the proof of Lemma 5.3.

The final step in the proof of Theorem 1.9. Let $Y := (Y_k, k \in \mathbf{Z}^d)$, $Z := (Z_k, k \in \mathbf{Z}^d)$, $U := (U_k, k \in \mathbf{Z}^d)$, and $\zeta := (\zeta_k, k \in \mathbf{Z}^d)$, be four (strictly stationary) random fields which are independent of each other, such that (i) the random field Y resp. Z resp. U has the properties specified in Lemma 5.1 resp. Lemma 5.2 resp. Lemma 5.3, and (ii) the random variables $\zeta_k,\ k \in \mathbf{Z}^d$ are independent (and identically distributed) with each being uniformly distributed on the interval $[0, 1]$. Referring to Definition 2.3(b), define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d, \quad X_k := \phi_4(Y_k, Z_k, U_k, \zeta_k) \ . \tag{5.43}$$

By an elementary argument, this random field X is strictly stationary. Also,

$$\forall k \in \mathbf{Z}^d, \quad \sigma(X_k) = \sigma(Y_k, Z_k, U_k, \zeta_k). \tag{5.44}$$

Also, for each element $q := (q_1, q_2, q_3, q_4) \in \mathbf{R}^4$, by Definition 2.3(b), $P(X_{\mathbf{0}} = \phi_4(q)) \leq P(\zeta_{\mathbf{0}} = q_4) = 0$ (and hence equality holds), and thus $P(X_{\mathbf{0}} = r) = 0$ for every real number r. Hence the distribution of $X_{\mathbf{0}}$ is completely nonatomic.

(*Remark:* The sole purpose of the random field ζ was to insure that the distribution of the random variable $X_{\mathbf{0}}$ would be nonatomic. However, that could also have been accomplished with a careful examination of the constructions in Lemma 4.2 and Lemma 5.3, eliminating the need for the random field ζ. A random field such as ζ apparently would be needed for this purpose in the particular extension of Theorem 1.9 described in the first two sentences of Remark 1.10(c). We shall not bother further with that here.)

By (5.44) and Lemma 2.1, for any positive integer n and two nonempty, disjoint sets $S, T \subset \mathbf{Z}^d$,

$$\begin{aligned} &\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \\ &\quad = \ \max\{\rho(\sigma(Y_k, k \in S), \sigma(Y_k, k \in T)),\ \rho(\sigma(Z_k, k \in S), \sigma(Z_k, k \in T)), \\ &\qquad\qquad \rho(\sigma(U_k, k \in S), \sigma(U_k, k \in T)),\ \rho(\sigma(\zeta_k, k \in S), \sigma(\zeta_k, k \in T))\} \ . \end{aligned}$$

Of course for any two nonempty, disjoint sets $S, T \subset \mathbf{Z}^d$, $\rho(\sigma(\zeta_k, k \in S), \sigma(\zeta_k, k \in T)) = 0$. Hence for any positive integer n,

$$\rho(X,n) = \max\{\rho(Y,n),\, \rho(Z,n),\, \rho(U,n)\}; \tag{5.45}$$

$$\rho'(X,n) = \max\{\rho'(Y,n),\, \rho'(Z,n),\, \rho'(U,n)\}; \quad \text{and} \tag{5.46}$$

$$\rho_1^*(X,n) = \max\{\rho_1^*(Y,n),\, \rho_1^*(Z,n),\, \rho_1^*(U,n)\}. \tag{5.47}$$

Now let verify the properties stated in Theorem 1.9 for the random field X here. It was already noted above that this random field X is strictly stationary, and that the distribution of $X_{\mathbf{0}}$ is nonatomic. Eq. (1.23) holds by (5.45), (5.6), (5.25), and (5.35). Eq. (1.25) holds by (5.46), (5.6), (5.26), (5.35), and (5.5). Eq. (1.27) holds by (5.47), (5.7), (5.26), (5.35), and (5.5).

Next, by (5.8), (5.44), and (1.27) (just proved above) for any given positive integer n and every ordered pair $(u,v) \in \{1,2,\dots,d\}^2$ such that $u < v$, one has that

$$c_n = \rho(\sigma(Y_{n\mathbf{e}(u)}, Y_{n\mathbf{e}(v)}), \sigma(Y_{\mathbf{0}})) \le \rho(\sigma(X_{n\mathbf{e}(u)}, X_{n\mathbf{e}(v)}), \sigma(X_{\mathbf{0}})) \le \rho_1^*(X,n) = c_n ,$$

which of course forces equality everywhere there. Thus (1.28) holds.

Eq. (1.26) holds by an analogous argument using (5.27), (5.44), and (1.25) (just proved above); and eq. (1.24) holds similarly by (5.36), (5.44), (1.8), and (1.23) (just proved above). That completes the proof of Theorem 1.9.

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