Metric discrepancy results for sequences with bounded gaps

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In case when $\{n_k\}$ satisfies Hadamard's gap condition $n_{k+1}/n_k > q > 1$, Walter Philipp proved

$$\frac{1}{4} < \varlimsup_{N\to\infty} \frac{ND_N(\{n_k x\})}{\sqrt{N \log\log N}} \le C_q < \infty \quad \text{a.e.}$$

and solved the Erdős-Gál conjecture.

This limsup equals to $1/\sqrt{2}$ when $n_{k+1}/n_k \to \infty$. Roughly saying, the sequence $\{\langle n_k x\rangle\}$ imitates the behaviour of uniform distributed i.i.d. when n_k diverges very rapidly.

Here, we consider the question of opposite direction, i.e., if there exists a sequence $\{n_k\}$ with $n_{k+1} - n_k = O(1)$ such that $\{\langle n_k x\rangle\}$ behaves like i.i.d.?

We know the following result by Berkes: for any $a_k \uparrow \infty$, there exists $\{n_k\}$ with $1 \le n_{k+1} - n_k \le 2a_k$ such that the CLT

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \cos 2\pi n_k x \xrightarrow{\mathcal{D}} \mathfrak{N}_{0,1/2}$$

holds. On the contrary, Bobkov-Götze proved that the above CLT does not hold under $n_{k+1} - n_k = O(1)$.

Relating to these results, we report the following: for any $0 < v < 1/2$, there exists a sequence $\{n_k\}$ of positive integers with

$$1 \le n_{k+1} - n_k \le \left\lceil (1+2v)^2/(1-2v)^2 \right\rceil$$

satisfying

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \cos 2\pi n_k x \xrightarrow{\mathcal{D}} \mathfrak{N}_{0,v} \quad \text{and} \quad \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{N \log\log N}} = \sqrt{v} \quad \text{a.e.}$$