The law of the iterated logarithm and the central limit theorem for systems $(f(n_k x))_{k\geq 1}$

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Consider a measurable function $f(x)$ on the space $([0,1], \mathcal{B}([0,1]), \lambda_{[0,1]})$ and a sequence $(n_k)_{k\geq 1}$ of positive integers. For special choices of f and (n_k), the system $(f(\langle n_k x\rangle))_{k\geq 1}$, where $\langle\cdot\rangle$ denotes the fractional part, forms a system of i.i.d. random variables, e.g. if $f(x) = \mathbf{1}_{[0,1/2]}(x) - 1/2$ and $n_k = 2^k,\ k \geq 1$. In general this does not have to be the case, but by the classical theory $(f(n_k x))_{k\geq 1}$ will "almost" behave like a system of i.i.d. random variables, provided f is a "nice" function and the sequence $(n_k)_{k\geq 1}$ is growing "fast". Typical results are the following, established by Erdős and Gál (1955) and Salem and Zygmund (1947), respectively: Assume that $(n_k)_{k\geq 1}$ satisfies the "Hadamard gap condition"

$$n_{k+1}/n_k > q > 1, \quad k \geq 1.$$

Then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos(2\pi n_k x)}{\sqrt{2N \log\log N}} = \frac{1}{\sqrt{2}} \quad \text{a.s.}$$

and

$$\lim_{N\to\infty} \lambda_{[0,1]} \left\{ x \in [0,1] : \sum_{k=1}^N \cos(2\pi n_k x) > t\sqrt{N/2} \right\} = \Phi(t), \qquad t \in \mathbb{R}.$$

If f is a more complicated function, e.g. a trigonometric polynomial, no precise results have been known for long time. Very recently, application of martingale approximation techniques has given rise to many interesting results. We show how the asymptotic behavior of the system $(f(n_k x))_{k\geq 1}$ is connected with the number of solutions of Diophantine equations of the form

$$a n_k \pm b n_l = c, \qquad a, b, c \in \mathbb{Z}.$$