A NOTE ON GENERATING THE FACTORS FOR SYMMETRY PARAMETERS IN LOG-LINEAR MODELS

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1 Introduction

The aim of this paper is to facilitate the fitting of symmetry and quasi-symmetry in two- and higher-dimensional log-linear models. A symmetry model for a square table is defined by

$$E(Y_{ij}) = \mu_{ij} = \exp(\beta^0 + \beta_i^A + \beta_j^B + \beta_{ij}^{AB}) \ , \tag{1}$$

where both classifications have the same single-factor parameters, i.e. $\beta_i^A = \beta_j^B$ if $i = j$ $(i = 1, \ldots, I,\ j = 1, \ldots, I)$, and all $\beta_{ij}^{AB} = \beta_{ji}^{AB}$. The residual degrees of freedom are $I(I-1)/2$.

Since the symmetry model is highly structured and implies marginal homogeneity it rarely provides a good fit to data. Relaxing the assumption of the single-factor parameterisation $\beta_i^A = \beta_i^B$, model (1) becomes the *quasi-symmetry model*, where $\beta_{ij}^{AB} = \beta_{ji}^{AB}$ for all $i \neq j$. As opposed to the symmetry model, there are no restrictions on the factors A and B, i.e. marginal homogeneity is not implied. The quasi-symmetry model has $(I-1)(I-2)/2$ residual degrees of freedom and applies to 3×3 tables and larger. For illustratitive purposes, the factor levels of the two-way interaction parameters in a 5×5-table may be given as

	B				
	1	2	4	7	11
	2	3	5	8	12
A	4	5	6	9	13
	7	8	9	10	14
	11	12	13	14	15

In three-way contingency tables quasi-symmetry is reflected by

$$\beta_{ijk}^{ABC} = \beta_{ikj}^{ABC} = \beta_{jik}^{ABC} = \beta_{jki}^{ABC} = \beta_{kij}^{ABC} = \beta_{kji}^{ABC} \ , \tag{2}$$

where no assumptions are made about the two-way interactions and the main effects. The table below gives an example for the factor levels for the symmetry parameters in a $3 \times 3 \times 3$ classification.

C_3:

C_1:

	B		
	1	2	5
A	2	3	6
	5	6	8

C_2:

	B		
	2	3	6
A	3	4	7
	6	7	9

C_3:

	B		
	5	6	8
A	6	7	9
	8	9	10

For a detailed presentation of symmetry models see Agresti (1990, Chapters 10 and 11). In the GLIM Newsletter 22, Kateri (1993) gave a collection of macros to fit various symmetry and asymmetry models to two-way classifications. Wheras Kateri's macros create all necessary terms for these models and fit

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them, the idea presented in this paper allows only for calculating and defining the symmetry structures, though for tables of arbitrary dimension. All model fitting is left to the user.

2 On the generation of factor levels

Wheras it is quite easy to calculate the factor levels for the symmetry parameters in two-way classifications, it is not so obvious in tables with three or a higher number of dimensions. Such models often occur in a latent variable context, cf. Hatzinger and Katzenbeisser (1994). In the special case of 2^L-tables the factor levels for the symmetry parameters are obtained by simply summing up the factor levels m_ℓ, $\ell = 1, \ldots, L$ of the L main effect terms represented by $m_l \in \{1, 2\}$, i.e.

$$s(\mathbf{m}) = 1 + \sum_{\ell=1}^{L} (m_\ell - 1) \ , \tag{3}$$

with $\mathbf{m} = (m_1, \ldots, m_L)$, indexing a certain cell in the contingency table. The corresponding GLIM-code for a 2^4-table might be

```
$GFAC A 2 B 2 C 2 D 2 $CA S=1+(A-1)+(B-1)+(C-1)+(D-1) $FAC S 5$ .
```

In general, for tables of size M^L, where $M > 2$ refers to the number of categories and L to the dimensionality, a simple sum like (3) does not yield a unique representation. This might be achieved by slightly modifying (3), i.e.

$$s^*(\mathbf{m}) = \sum_{\ell=1}^{L} L^{(m_\ell - 1)} \ . \tag{4}$$

A proof of the uniqueness in the sense $s^*(\mathbf{m}) \neq \mathbf{s}^*(\mathbf{n})$ for $\mathbf{m} \neq \mathbf{n}$ is given in the Appendix.

For a 3^4-table the GLIM-code to calculate s^* is

```
$GFAC A 3 B 3 C 3 D 3 $
$CA S=4**(A-1)+4**(B-1)+4**(C-1)+4**(D-1) $ ,
```

according to (4).

However, the maximum number of s^* is L^M, wheras only $\binom{L+M-1}{M-1}$ levels are needed. To avoid unnecessary aliasing by fitting a factor with L^M levels, the `$GROUP`–directive should be used:

```
$GROUP S $
```

3 Example

Hand et.al. (1993) present a data set on three aspects of colouring of blackbirds. The colours of the orbital ring (OR), the upper mandile (UM), and the lower mandile (LM) were recorded as 1 to 3 in 90 birds. The question was whether the colours of these three features were related. The main extracts of a transcript file are given in the following.

```
[e] ! Macro SYM_PAR is already input
[e] !
[e] ! Dataset given by Hand,et.al, Small Data Sets, No 224
[e] !
[e] $SL 27
[e] $READ Y
[e] 40 1 1  19 6 2  0 0 0
[e]  0 1 0   0 2 1  0 1 1
[e]  0 0 0   1 1 6  0 0 7
[e] $GFAC OR 3 UM 3 LM 3$
[e] $CA S=3**(OR-1)+3**(UM-1)+3**(LM-1)
[e] $GROUP S
[e] $ERR P$YVAR Y$CYCLE 20
[e] !
[e] $F S        ! complete symmetry
[o] scaled deviance =  64.413 at cycle 7
[o]     residual df =  17
[e] !
[e] $F +OR+LM   ! quasi symmetry
[e] $D E$
[o] scaled deviance =  12.617 (change =   -51.80) at cycle 9
[o]     residual df =  13     (change =    -4   )
[o]
[o]           estimate        s.e.     parameter
[o]      1       3.689      0.1581      1
[o]      2     -0.8115      0.2833      S(2)
[o]      3      0.3152      0.7774      S(3)
[o]      4       3.205       1.559      S(4)
[o]      5      -4.549       1.163      S(5)
[o]      6      -3.016       1.122      S(6)
[o]      7      0.5703       1.612      S(7)
[o]      8      -12.56       40.99      S(8)
[o]      9     -0.3077       1.934      S(9)
[o]     10     -0.1218       2.028      S(10)
[o]     11      -3.963      0.9092      OR(2)
[o]     12      -1.803       1.206      OR(3)
[o]     13      -2.238      0.6709      LM(2)
[o]     14      0.1820       1.023      LM(3)
[o] scale parameter 1.000
```

4 Appendix

Without loss of generality let $\mathbf{m} = m_1, \ldots, m_L$ be ordered as $1 \leq m_1 \leq m_2 \leq \ldots \leq m_L \leq M$ and $\mathbf{n} = n_1, \ldots, n_L$ denote a different vector of ordered arguments. We would like to prove that if $\mathbf{m}$ and $\mathbf{n}$ differs in some components the respective values of s^* are also different.

Therefore, we first assume that there is already a difference in the largest value of the vectors, i.e. $m_L > n_L$ implying $m_L - 1 \geq n_\ell, \forall \ell = 1, \ldots, L$. This yields

$$s^*(\mathbf{n}) = \sum_{\ell=1}^{L} L^{n_\ell - 1} \leq \sum_{\ell=1}^{L} L^{(m_L - 1) - 1} = L^{m_L - 1} < \sum_{\ell=1}^{L} L^{m_\ell - 1} = s^*(\mathbf{m}).$$

More generally, let the K largest arguments of $\mathbf{m}$ and $\mathbf{n}$ be equal, that means $m_{L+1-k} = n_{L+1-k}, \forall k = 1, \ldots, K < L$, and suppose $m_{L-K} > n_{L-K}$ or equivalentely $m_{L-K} - 1 \geq n_\ell, \forall \ell = 1, \ldots, L-K$. Hence,

$$
\begin{aligned}
s^*(\mathbf{n}) &= \sum_{\ell=L+1-K}^{L} L^{m_\ell - 1} + \sum_{\ell=1}^{L-K} L^{n_\ell - 1} \leq \sum_{\ell=L+1-K}^{L} L^{m_\ell - 1} + \sum_{\ell=1}^{L-K} L^{(m_{L-K}-1)-1} \\
&< \sum_{\ell=L+1-K}^{L} L^{m_\ell - 1} + L^{m_{L-K}-1} < \sum_{\ell=L+1-K}^{L} L^{m_\ell - 1} + \sum_{\ell=1}^{L-K} L^{m_\ell - 1} = s^*(\mathbf{m})
\end{aligned}
$$

□

Alternatively, mapping (4) can be interpreted as

$$
s^*(\mathbf{m}) = s^*(\mathbf{c}) = \sum_{i=1}^{M} c_i L^{i-1}, \tag{5}
$$

where the element c_i of $c = c_1, \ldots, c_M$ is a counter describing how many arguments m_ℓ equals $M+1-i$. Therefore, the representation (5) enables us to consider the mapping (4) as as a transformation of the L-adic number $c_1 \ldots c_M$ to the corresponding decadic number $s^*(\mathbf{c})$ and because of $s^*(\mathbf{m}) = s^*(\mathbf{c})$ the uniqueness of the transformation (4) follows immediately.

5 References

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