Flexible Regression and Smoothing

Discrete Distributions

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Count distributions

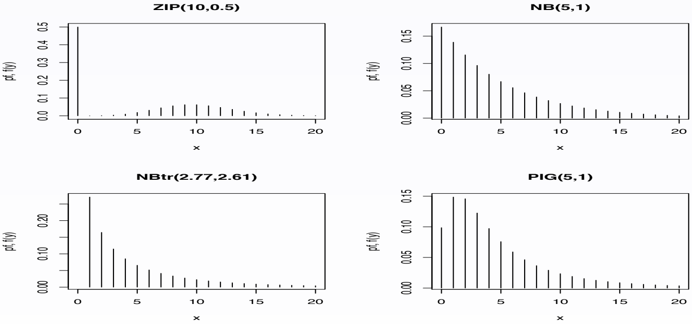
The three major problems encounter when modelling count data using the Poisson distribution.

- overdispersion
- excess (or shortage) of zero values
- long tails (rare events)

Discrete distribution modelling

Par.	Modelling	Distributions
1	Location	PO
2	Location and scale	NBI, NBII, PIG
2	Location and zero probability	ZALG, ZAP, ZIP, ZIP2
3	Location, scale and skewness	DEL, SI, SICHEL
3	Location, scale and zero probability	ZANBI, ZINBI, ZIPIG

Different count data distributions



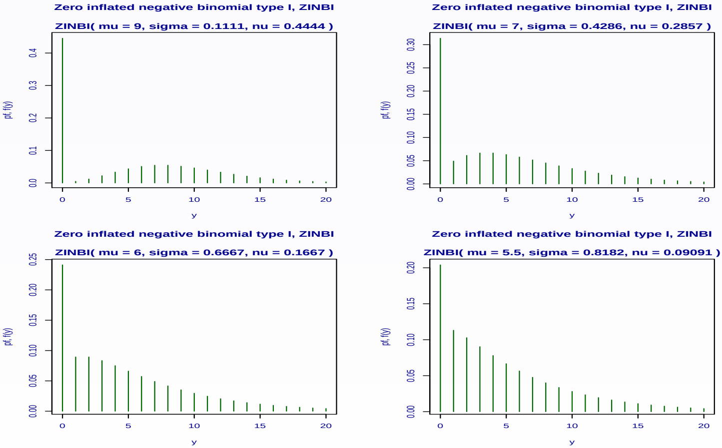
Zero inflated distributions

Zero inflated distribution, $Y \sim \mathbf{ZID}$ is given by
$Y = 0$ with probability p
$Y \sim \mathbf{D}$ with probability $1 - p$.
Hence

$$P(Y = y) = \begin{cases} p + (1-p)P(Y_1 = 0) & \text{if } y = 0 \\ (1-p)P(Y_1 = y) & \text{if } y = 1, 2, 3, \ldots \end{cases}$$

where $Y_1 \sim \mathbf{D}$.

ZINBI distribution plots



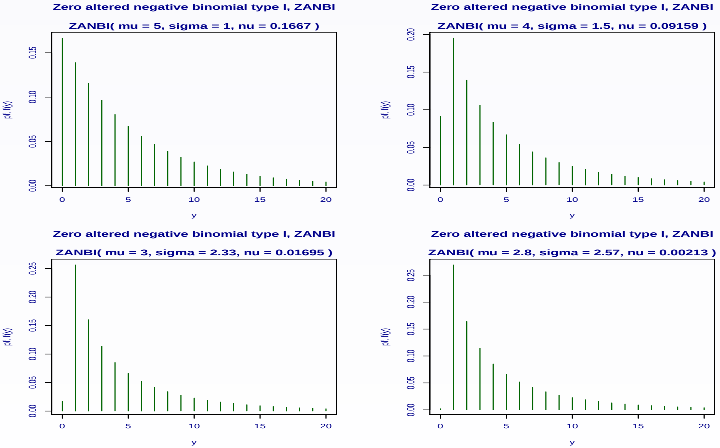
Zero adjusted distributions

Zero adjusted distribution, $Y \sim \textbf{ZAD}$ is given by
$Y = 0$ with probability p
$Y \sim \textbf{Dtr}$ with probability $1 - p$,
where **Dtr** is a truncated distribution, **D** truncated at zero.
Hence

$$P(Y = y) = \begin{cases} p & \text{if } y = 0 \\ (1-p)\frac{P(Y_1 = y)}{1 - P(Y_1 = 0)} & \text{if } y = 1, 2, 3, \ldots \end{cases} \tag{1}$$

where $Y_1 \sim \textbf{D}$.

ZANBI distribution plots



Different (overdispersed) count data approaches

(a) *Ad-hoc* solutions
  (i) quasi-likelihood (QL), Extended QL
  (ii) Efron's Double Exponential
  (iii) pseudo-likelihood (PL)

(b) Discretized continuous distributions
for example if $F_W(w)$ is the cdf a continuous random variable W defined in $\Re^+$ then $f_Y(y) = F_W(y+1) - F_W(y)$

(c) Random effect at the observation level solutions.
$f_Y(y) = \int f(y|\gamma) f_\gamma(\gamma) d\gamma$.

(c) Random effect at the observation level

(i) when an an explicit continuous mixture distribution, $f_Y(y)$, exists.

(ii) when a continuous mixture distribution, $f_Y(y)$, is not explicit but is approximated by integrating out the random effect using approximations, e.g. Gaussian quadrature or Laplace approximation.

(iii) when a 'non-parametric' mixture (effectively a finite mixture) is assumed for the response variable.

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Random effect at the observation level case (i)

(i) Explicit continuous mixture distribution

$$\underbrace{f_Y(y)}_{\text{discrete}} = \int \underbrace{f(y|\gamma)}_{\text{discrete}} \underbrace{f_\gamma(\gamma)}_{\text{continuous}} \, d\gamma$$

- $Y \sim NBI(\mu, \sigma)$
- $Y|\gamma \sim PO(\gamma\mu)$
- $\gamma \sim GA(1, \sigma^{1/2})$

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Random effect at the observation level case (ii)

(ii) Non-explicit continuous mixture distribution

$$\underbrace{f_Y(y)}_{\text{discrete}} = \int \underbrace{f(y|\gamma)}_{\text{discrete}} \underbrace{f_\gamma(\gamma)}_{\text{continuous}} \, d\gamma$$

- $Y \sim PO - Normal(\mu, \sigma)$
- $Y|\gamma \sim PO(\gamma\mu)$
- $\log(\gamma) \sim NO(1, \sigma)$

Random effect at the observation level case (iii)

(iii) Non-parametric mixture distribution

$$\underbrace{f_Y(y)}_{\text{discrete}} = \sum_{k=1}^{K} \underbrace{f(y|\gamma_k)}_{\text{discrete}} \underbrace{p(\gamma = \gamma_k)}_{\text{continuous}}$$

- $Y \sim PO - NPFM(\mu, \sigma)$
- $Y|\gamma \sim PO(\gamma\mu)$
- $\log(\gamma) \sim NPFM(2)$

where NPFM(2) equals Non-Parametric Finite Mixture with 2 point probabilities

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Explicit continuous mixture distribution

Distributions	R Name	mixing distribution for γ
Poisson	$\texttt{PO}(\mu)$	-
Neg. bin. I	$\texttt{NBI}(\mu, \sigma)$	$\texttt{GA}(1, \sigma^{\frac{1}{2}})$
Neg. bin. II	$\texttt{NBII}(\mu, \sigma)$	$\texttt{GA}(1, \sigma^{\frac{1}{2}}/\mu)$
Poisson IG	$\texttt{PIG}(\mu, \sigma)$	$\texttt{IG}(1, \sigma^{\frac{1}{2}})$
Sichel	$\texttt{SICHEL}(\mu, \sigma, \nu)$	$\texttt{GIG}(1, \sigma^{\frac{1}{2}}, \nu)$
Delaporte	$\texttt{DEL}(\mu, \sigma, \nu)$	$\texttt{SG}(1, \sigma^{\frac{1}{2}}, \nu)$
Zero inflated Poisson	$\texttt{ZIP}(\mu, \sigma)$	$\texttt{BI}(1, 1-\sigma)$
Zero inflated Poisson 2	$\texttt{ZIP2}(\mu, \sigma)$	$(1-\sigma)^{-1}\texttt{BI}(1, 1-\sigma)$
Zero inflated neg. bin.	$\texttt{ZINBI}(\mu, \sigma, \nu)$	zero inflated gamma
Poisson-Tweedie	-	Tweedie family

Table: Discrete gamlss family distributions for count data

R Name	params	mean	variance
$\texttt{PO}(\mu)$	1	μ	μ
$\texttt{NBI}(\mu, \sigma)$	2	μ	$\mu + \sigma\mu^2$
$\texttt{NBII}(\mu, \sigma)$	2	μ	$\mu + \sigma\mu$
$\texttt{PIG}(\mu, \sigma)$	2	μ	$\mu + \sigma\mu^2$
$\texttt{SICHEL}(\mu, \sigma, \nu)$	3	μ	$\mu + h(\sigma, \nu)\mu^2$
$\texttt{DEL}(\mu, \sigma, \nu)$	3	μ	$\mu + \sigma(1-\nu)^2\mu^2$
$\texttt{ZIP}(\mu, \sigma)$	2	$(1-\sigma)\mu$	$(1-\sigma)\mu + \sigma(1-\sigma)\mu^2$
$\texttt{ZIP2}(\mu, \sigma)$	2	μ	$\mu + \frac{\sigma}{(1-\sigma)}\mu^2$

Families modelling the variance-mean relationship

$V[Y] = \mu + \mu^2 V[\gamma]$ where $V[\gamma] = \upsilon(\sigma, \nu, \tau)$ is a function of the parameters of the mixing distribution $f_\gamma(\gamma)$.
Alternative variance-mean relationship can be obtained by reparametrization.
i.e NB type I $V[Y] = \mu + \sigma\mu^2$.
If $\sigma = \sigma_1/\mu$ then
$V[Y] = (1 + \sigma_1)\mu$ (negative binomial type II)
$\sigma = \sigma_1\mu$ then $V[Y] = \mu + \sigma_1\mu^3$.
More generally $\sigma = \sigma_1\mu^{2-\nu}$ giving $V(Y) = \mu + \sigma_1\mu^\nu$

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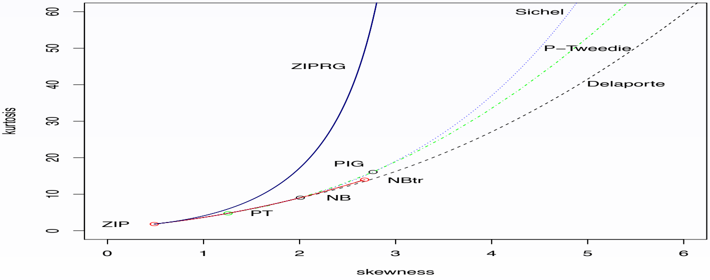
Comparison of the marginal distributions using a (ratio moment) diagram of their skewness and kurtosis



A stylometric application

Data summary:

R data file: `stylo` in package **gamlss.data** of dimensions 64×2

source: Dr Mario Corina-Borja

variables

`word` : is the number of times a word appears in a single text

`freq` : the frequency of the number of times a word appears in a text

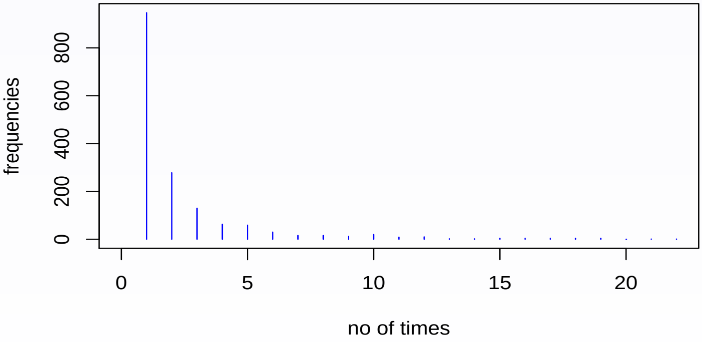
purpose: to demonstrate the fitting of a truncated discrete dist.

conclusion the truncated SICHEL distributions fits best

A stylometric application

```
 library(gamlss.tr)
 data(stylo)
 plot(freq ~ word, data = stylo, type = "h", xlim =
+  c(0, 22), xlab = "no of times", ylab =
+ "frequencies", col = "blue")
```

The stylometric data



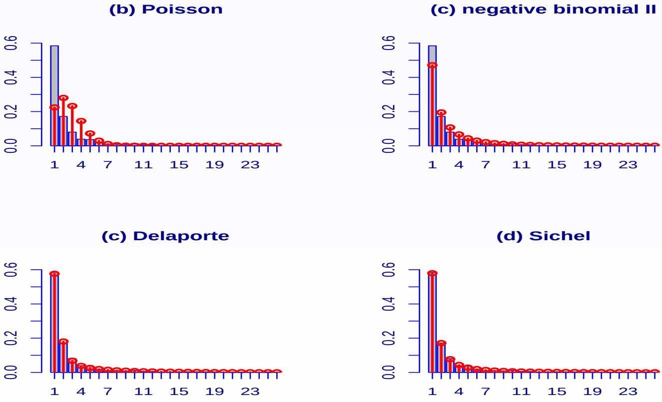
A stylometric application

```
> library(gamlss.tr)
> gen.trun(par = 0, family = PO, type = "left")
A truncated family of distributions from PO has been generated
 and saved under the names:
 dPOtr pPOtr qPOtr rPOtr POtr
The type of truncation is left and the truncation parameter is
> gen.trun(par = 0, family = NBII, type = "left")
...
> gen.trun(par = 0, family = DEL, type = "left")
...
> gen.trun(par = 0, family = SICHEL, type = "left",
+       delta = 0.001)
...
```

A stylometric application

```
> mPO <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = POtr, trace = FALSE)
> mNBII <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = NBIItr, n.cyc = 50, trace = FALSE)
> mDEL <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = DELtr, n.cyc = 50, trace = FALSE)
> mSI <- gamlss(word ~ 1, weights = freq, data = stylo,
+ family = SICHELtr, n.cyc = 50, trace = FALSE)
> GAIC(mPO, mNBII, mDEL, mSI)
      df      AIC
mSI    3 5148.454
mDEL   3 5160.581
mNBII  2 5311.627
mPO    1 9207.459
```

The stylometric data



The fish species data

Data summary: the fish species data

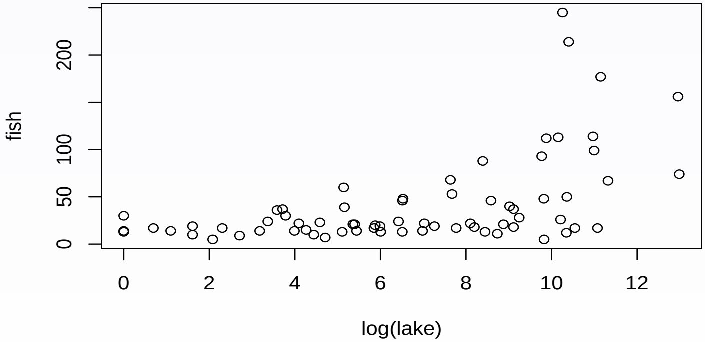
R data file: `species` in package gamlss.data of dimensions 70×2

variables

`fish` : the number of different species in 70 lakes in the world

`lake` : the lake area

The fish species data



The fish species data

There are several questions that need to be answered.

- How does the mean of y depend on x?
- Is y overdispersed Poisson?
- How does the variance y depend on its mean?
- What is the distribution of y given x?
- Do the scale and shape parameters of the distribution of y depend on x?

Overdispersed count data approaches

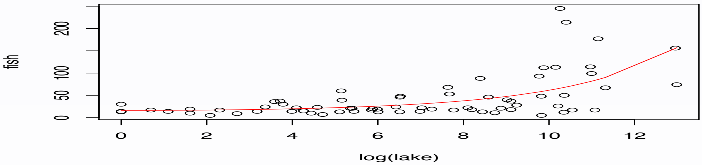
Table: Comparison of models for the fish species data

Model	$f_Y(y)$	μ	σ	ν	DEV	df	AIC	SBC
1	PO	$x<2>$	-	-	1849.3	3	1855.3	1862.0
2	NBI	x	1	-	619.8	3	625.8	632.6
3	NBI	$x<2>$	1	-	614.3	4	622.3	631.3
4	NBI	$cs(x,3)$	1	-	611.9	6	623.9	637.4
5	NBI	$x<2>$	x	-	605.0	5	615.0	626.2
6	NBI-fam	$x<2>$	1	1	606.0	5	616.0	627.3
7	NBI-fam	$x<2>$	x	1	604.9	6	616.9	630.4

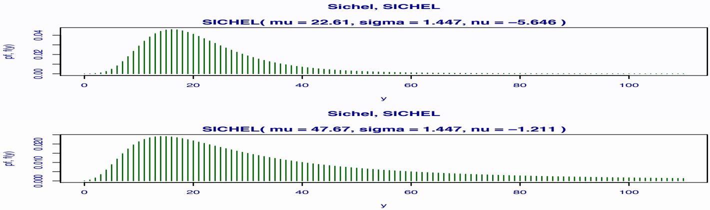
Overdispersed count data approaches

Model	$f_Y(y)$	μ	σ	ν	DEV	df	AIC	SBC
8	PIG	$x < 2 >$	1	-	613.3	4	621.3	630.3
9	SI	$x < 2 >$	1	x	597.7	6	609.7	623.2
10	DEL	$x < 2 >$	1	x	600.6	6	612.6	626.1
11	DEL	$x < 2 >$	-	x	600.6	5	610.6	621.9
12	PO-Normal	$x < 2 >$	1	-	615.2	4	623.2	632.2
13	NBI-Normal	$x < 2 >$	x	1	603.7	6	615.7	629.2
14	PO-NPFM(5)	$x < 2 >$	-	–	601.9	13	627.9	657.2
15	NB-NPFM(2)	$x < 2 >$	1	–	611.9	6	623.9	637.4
16	doublePO	$x < 2 >$	x	-	616.4	5	626.4	637.6
17	IGdisc	$x < 2 >$	1	-	603.3	4	611.3	620.3

Fitted mean of the Sichel distribution



Fitted Sichel distributions for observations (a) 40 and (b) 67



Binomial response variables

There are only two distributions here

- binomial
- beta binomial

The hospital stay data

Data summary:

R data file: aep in package **gamlss** of dimensions 1383×8

source: Gange *et al.* (1996)

variables

`los` : total number of days

`noinap` : number of inappropriate days patient stay in hospital

`loglos` : the log of `los/10`

`sex` : the gender of patient

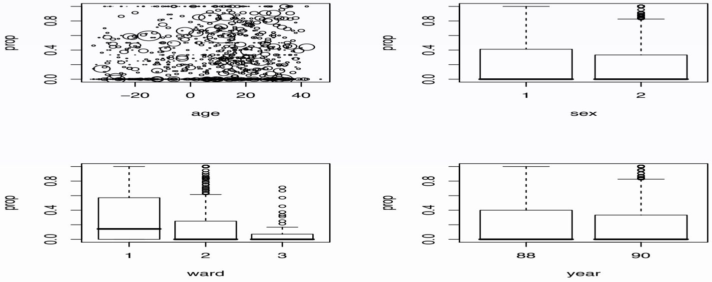
`ward` : type of ward in the hospital (medical, surgical or

`year` : 1988 or 1990

`age` : age of the patient subtracted from 55

`y` : the response variable, a matrix with columns (`noinap`, `los-noinap`)

The hospital stay data



The hospital stay data

```
> mI <- gamlss(y ~ ward + year + loglos, sigma.fo = ~year,
+ family = BB,data = aep)
> mII <- gamlss(y ~ ward + year + loglos, sigma.fo = ~year +
+ ward, family = BB, data = aep)
> mIII <- gamlss(y ~ ward + year + cs(loglos, 1),
+ sigma.fo = ~year+ward, family = BB, data = aep)
> mIV <- gamlss(y ~ ward + year + cs(loglos, 1) + cs(age, 1),
+ sigma.fo = ~year + ward, family = BB, data = aep)
> GAIC(mI, mII, mIII, mIV, k = 0)
           df      AIC
mIV  12.00010 4454.362
mIII 10.00045 4459.427
mII   9.00000 4483.020
mI    7.00000 4519.441
```

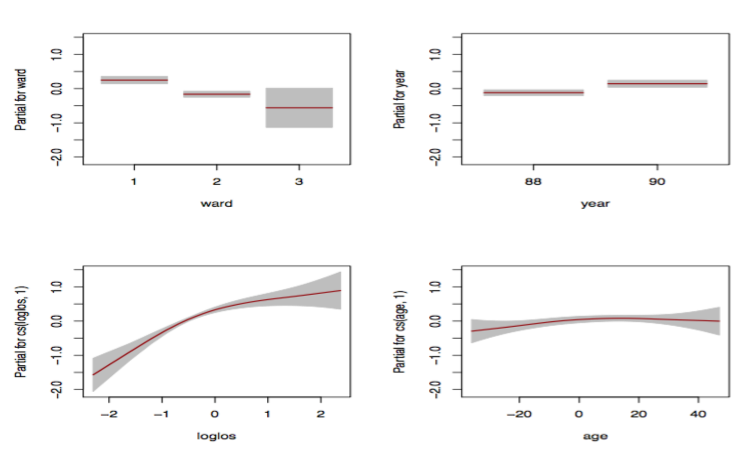
The hospital stay data

Models	Links	Terms	GD (AIC) [SBC]
I	$\text{logit}(\mu)$ $\log(\sigma)$	1+ward+loglos+year 1+year	4519.4 (4533.4) [4570.1]
II	$\text{logit}(\mu)$ $\log(\sigma)$	1+ward+loglos+year 1+year+ward	4483.0 (4501.0) [4548.1]
III	$\text{logit}(\mu)$ $\log(\sigma)$	1+ward+cs(loglos,1)+year 1+year+ward	4459.4 (4479.4) [4531.8]
IV	$\text{logit}(\mu)$ $\log(\sigma)$	1+ward+cs(loglos,1)+year+cs(age,1) 1+year+ward	4454.4 (4478.4) [4541.2]

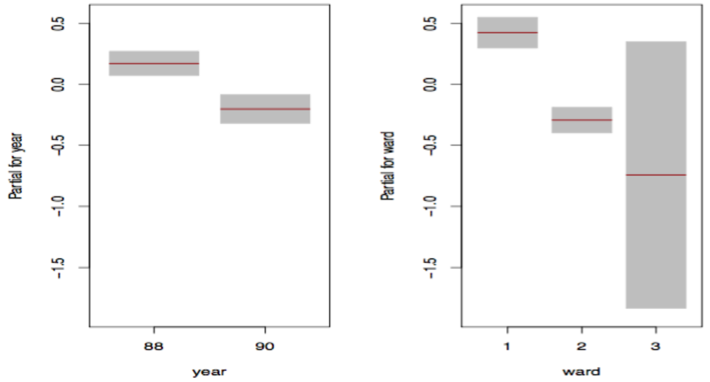
The hospital stay data

```
> op <- par(mfrow = c(2, 2))
> term.plot(mIV, se = T)
> par(op)
> op <- par(mfrow = c(2, 1))
> term.plot(mIV, "sigma", se = T)
> par(op)
> rqres.plot(mIV)
```

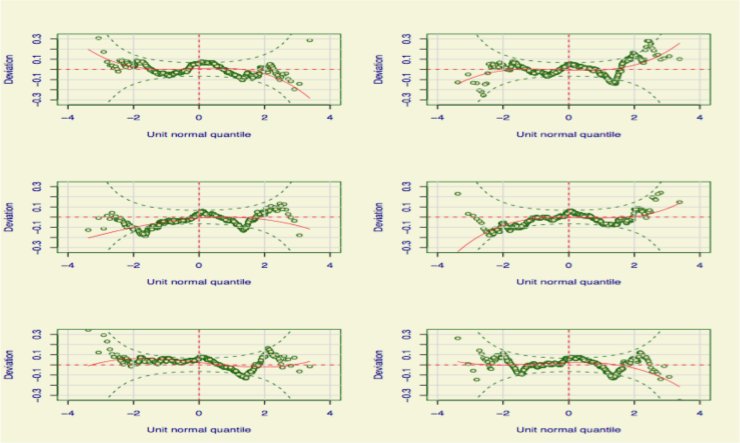
The hospital stay data: fitted model fop μ



The hospital stay data: fitted model for σ



The hospital stay data: normalised randomised quantile residuals



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for more information see

www.gamlss.org