# 2. Inference in Regression Analysis

If $Y_i \sim N(\mu_i, \sigma_i^2)$, $Y_i$'s are independent, and $a_1, \ldots, a_n$ are known constants then

$$\sum_{i=1}^{n} a_i Y_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Thus, a linear combination of independent normal random variables is itself a normal random variable.

**Theorem:** $b_0$ and $b_1$ are linear combinations of the $Y_i$'s. That is, we can write

$$b_1 = \sum_{i=1}^{n} k_i Y_i \quad \text{and} \quad b_0 = \sum_{i=1}^{n} l_i Y_i$$

where $k_1, \ldots, k_n$ and $l_1, \ldots, l_n$ are known constants.

**Proof:** Recall $S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2$. So

$$
\begin{aligned}
b_1 &= \frac{1}{S_{XX}} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) \\
&= \frac{1}{S_{XX}} \left[ \sum_{i=1}^{n} (X_i - \bar{X}) Y_i - \bar{Y} \sum_{i=1}^{n} (X_i - \bar{X}) \right] \\
&= \frac{1}{S_{XX}} \sum_{i=1}^{n} (X_i - \bar{X}) Y_i = \sum_{i=1}^{n} \left( \frac{X_i - \bar{X}}{S_{XX}} \right) Y_i \\
&= \sum_{i=1}^{n} k_i Y_i \quad \text{with} \quad k_i = \frac{X_i - \bar{X}}{S_{XX}}
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= \bar{Y} - b_1\bar{X} = \frac{1}{n}\sum_{i=1}^{n} Y_i - \bar{X}\sum_{i=1}^{n} k_i Y_i \\
&= \sum_{i=1}^{n}\left(\frac{1}{n} - k_i\bar{X}\right) Y_i \\
&= \sum_{i=1}^{n} l_i Y_i \quad \text{with} \quad l_i = \frac{1}{n} - k_i\bar{X}.
\end{aligned}
$$

Thus, $b_0$ and $b_1$ are linear combinations of the $Y_i$'s and, hence, they are normal variates. What about their means and variances?

**Theorem:** Under SLR model with normal errors:

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2}{S_{XX}}\right) \quad \text{and} \quad b_0 \sim N\left(\beta_0, \frac{\sigma^2}{n}\frac{\sum_i X_i^2}{S_{XX}}\right).$$

We are first interested in $\sum_i k_i$, $\sum_i k_i X_i$ and $\sum_i k_i^2$.

$$\sum_{i=1}^n k_i = \sum_{i=1}^n \frac{X_i - \bar{X}}{S_{XX}} = \frac{1}{S_{XX}} \sum_{i=1}^n (X_i - \bar{X}) = 0$$

$$\sum_{i=1}^n k_i X_i = \sum_{i=1}^n \frac{X_i - \bar{X}}{S_{XX}} X_i = \frac{1}{S_{XX}} S_{XX} = 1$$

$$\sum_{i=1}^n k_i^2 = \frac{1}{S_{XX}^2} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{S_{XX}}.$$

**Proof:** Since $b_1 = \sum_{i=1}^n k_i Y_i$, we get

$$\mathsf{E}(b_1) = \sum_{i=1}^n k_i \mathsf{E}(Y_i) = \sum_{i=1}^n k_i(\beta_0 + \beta_1 X_i).$$

Because $\sum_i k_i = 0$ and $\sum_i k_i X_i = 1$, this is

$$\mathsf{E}(b_1) = \beta_0 \sum_{i=1}^n k_i + \beta_1 \sum_{i=1}^n k_i X_i = \beta_1.$$

With $\sum_i k_i^2 = 1/S_{XX}$ , we get

$$\mathsf{var}(b_1) \quad = \quad Var\left(\sum_{i=1}^n k_i Y_i\right) = \sum_{i=1}^n k_i^2 Var(Y_i) = \sigma^2 \sum_{i=1}^n k_i^2 = \frac{\sigma^2}{S_{XX}}.$$

Showing $b_0 \sim N\left(\beta_0, \frac{\sigma^2}{n}\frac{\sum_i X_i^2}{S_{XX}}\right)$ is basically the same.

**Example:** 93 house prices in G'ville sold Dec. 1995.
$Y$ = selling price (in 1,000$), $X$ = area (1,000 sq.feet)

Assume the SLR model holds

$$\mathsf{E}(Y_i) = \beta_0 + \beta_1 X_i$$

LS estimators are $b_0 = -25.2$ and $b_1 = 75.6$. We are interested in testing

$H_0 : \beta_1 = 0$ (no linear relation between area and price) $H_A : \beta_1 \neq 0$

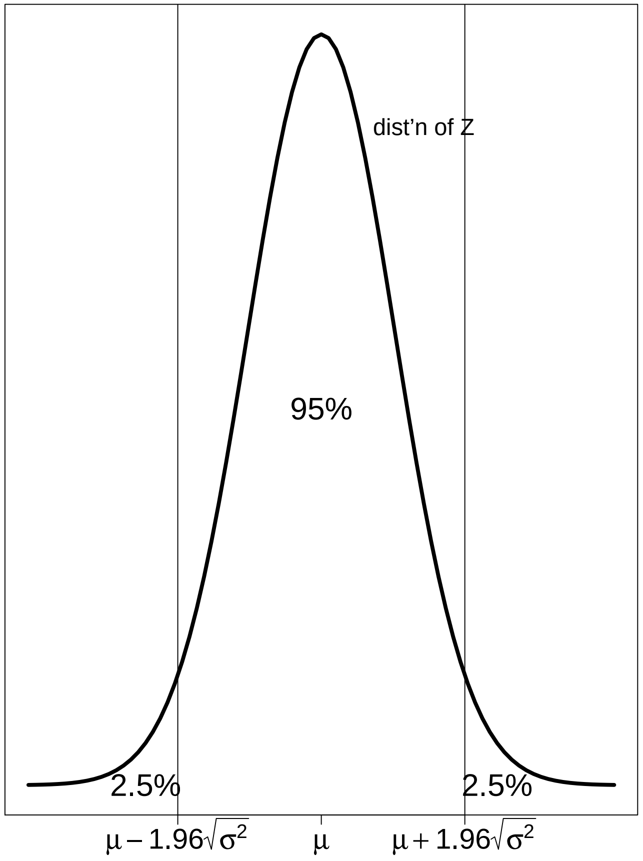
Since $75.6 \neq 0$, can we conclude that $H_A$ is true?

Recall: $b_1 \sim N(\beta_1, \sigma^2/S_{XX})$, where $S_{XX} = \sum_i (X_i - \bar{X})^2 = 25.38$.

Consider 2 different scenarios:

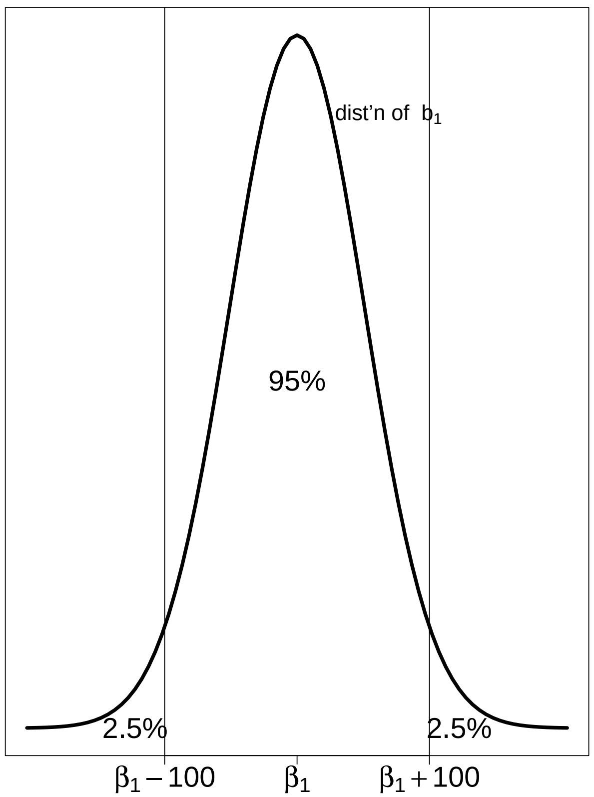
Scenario 1: $\sigma^2/S_{XX} = 2,500 \Rightarrow \sqrt{\sigma^2/S_{XX}} = 50$
Scenario 2: $\sigma^2/S_{XX} = 100 \Rightarrow \sqrt{\sigma^2/S_{XX}} = 10$

Remember, if $Z \sim N(\mu, \sigma^2)$, then

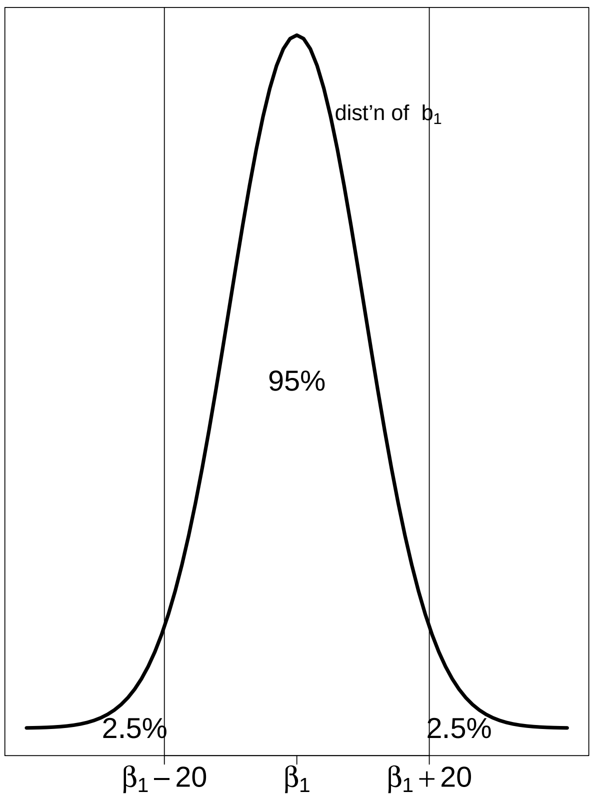


For the 2 scenarios we have:

Scenario 1: $\sqrt{\sigma^2/S_{XX}} = 50$




Scenario 2: $\sqrt{\sigma^2/S_{XX}} = 10$



**Scenario 1:** If $\beta_1 = 0$ ($H_0$ true) then there is a 95% chance that $b_1$ falls between $-100$ and 100.
$b_1 = 75.6$ is consistent with $H_0 : \beta_1 = 0$

**Scenario 2:** If $\beta_1 = 0$ ($H_0$ true) then there is a 95% chance that $b_1$ falls between $-20$ and 20.
$b_1 = 75.6$ suggests that $H_0 : \beta_1 = 0$ is false.

**Conclusion:** if we know $\sqrt{\sigma^2/S_{XX}}$, we know how likely the value $b_1 = 75.6$ is under $H_0$, and we can decide if $b_1 = 75.6$ is more consistent with $H_0 : \beta_1 = 0$ or $H_A : \beta_1 \neq 0$.

Last time we showed that

$$b_1 \sim N(\beta_1, \sigma^2/S_{XX}) \quad \Rightarrow \quad \frac{b_1 - \beta_1}{\sqrt{\sigma^2/S_{XX}}} \sim N(0,1)$$

That means that

$$P\left(-1.96 \leq \frac{b_1 - \beta_1}{\sqrt{\sigma^2 / S_{XX}}} \leq 1.96\right) = 0.95$$

$$P\left(b_1 - 1.96\sqrt{\frac{\sigma^2}{S_{XX}}} \leq \beta_1 \leq b_1 + 1.96\sqrt{\frac{\sigma^2}{S_{XX}}}\right) = 0.95$$

So, a **95% confidence interval** for $\beta_1$ is

$$b_1 \pm 1.96\sqrt{\frac{\sigma^2}{S_{XX}}}$$

Is this a useful confidence interval ? **NO!**

We have to estimate $\sigma^2$ under the SLR model. Remember, the mean squared error

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2 = \text{MSE}$$

is an unbiased estimate of $\sigma^2$. So we have all we need!

**What's next?**

1. tests and confidence intervals for $\beta_1$

2. confidence intervals for the mean of $Y$ at some value of $X$, say $X^*$, that is, for

$$\beta_0 + \beta_1 X^*$$

3. prediction intervals for the next random variable observed with $X = X^*$

# Confidence Intervals and Tests for $\beta_1$

The key is $b_1 \sim N(\beta_1, \sigma^2/S_{XX})$. Thus

$$\frac{b_1 - \beta_1}{\sqrt{\sigma^2/S_{XX}}} \sim N(0,1)$$

But this is not useful because we don't know $\sigma^2$.

If we replace $\sigma^2$ with our estimate of $\sigma^2$, MSE, we get

$$\frac{b_1 - \beta_1}{\sqrt{\mathsf{MSE}/S_{XX}}} \sim t(n-2).$$

Everything is based on this!

In what follows, $\alpha$ is:

- the type 1 error probability $=$ P(reject $H_0 \mid H_0$ true)
- always between 0 and 1 (it's a probability)
- usually set at 0.01, 0.05 or 0.10

## $(1-\alpha)100\%$ Confidence Interval for $\beta_1$

With probability $1-\alpha$

$$-t(1-\alpha/2; n-2) \leq \frac{b_1 - \beta_1}{\sqrt{\mathsf{MSE}/S_{XX}}} \leq t(1-\alpha/2; n-2)$$

Thus, the $(1-\alpha) * 100\%$ confidence interval for $\beta_1$ is

$$b_1 \pm t(1-\alpha/2; n-2)\sqrt{\mathsf{MSE}/S_{XX}}$$

Don't confuse $t(n-2)$ with $t(1-\alpha/2; n-2)$:

- $t(n-2)$: denotes the type of distribution ($t$) and its parameter ($n-2$)
- $t(1-\alpha/2; n-2)$: denotes the $1-\alpha/2$ percentile of the $t(n-2)$ distribution

## $\alpha$ Level Hypothesis Tests concerning $\beta_1$

**A** Two-Sided Test $H_0 : \beta_1 = c$, $H_A : \beta_1 \neq c$

**B** One-Sided Test $H_0 : \beta_1 \geq c$, $H_A : \beta_1 < c$

**C** One-Sided Test $H_0 : \beta_1 \leq c$, $H_A : \beta_1 > c$

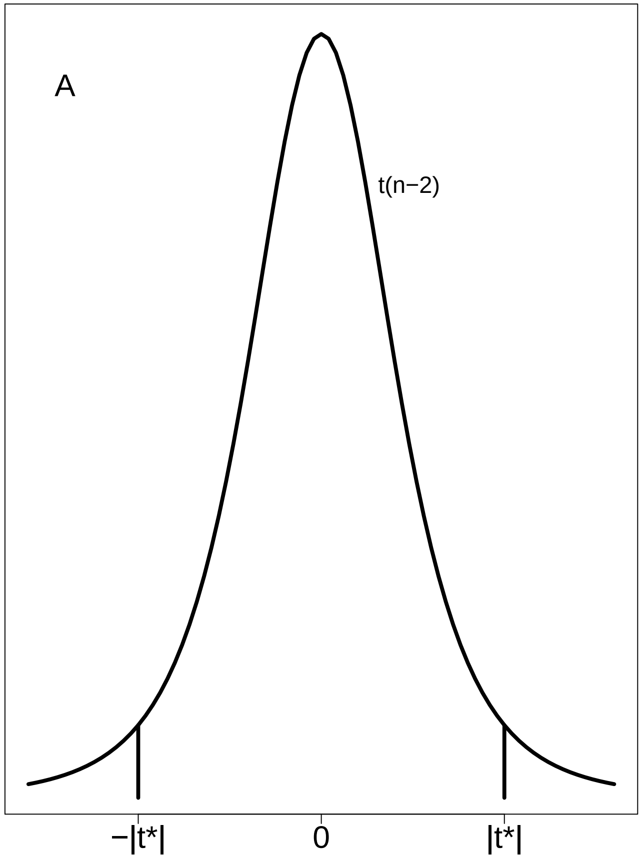
**Test Statistic:**

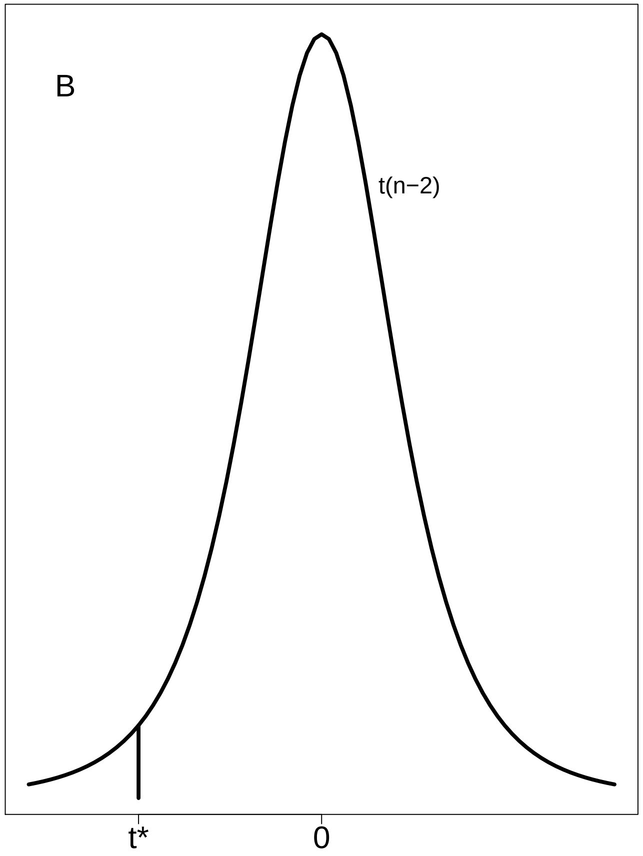
$$t^* = \frac{b_1 - c}{\sqrt{\mathsf{MSE}/S_{XX}}}$$

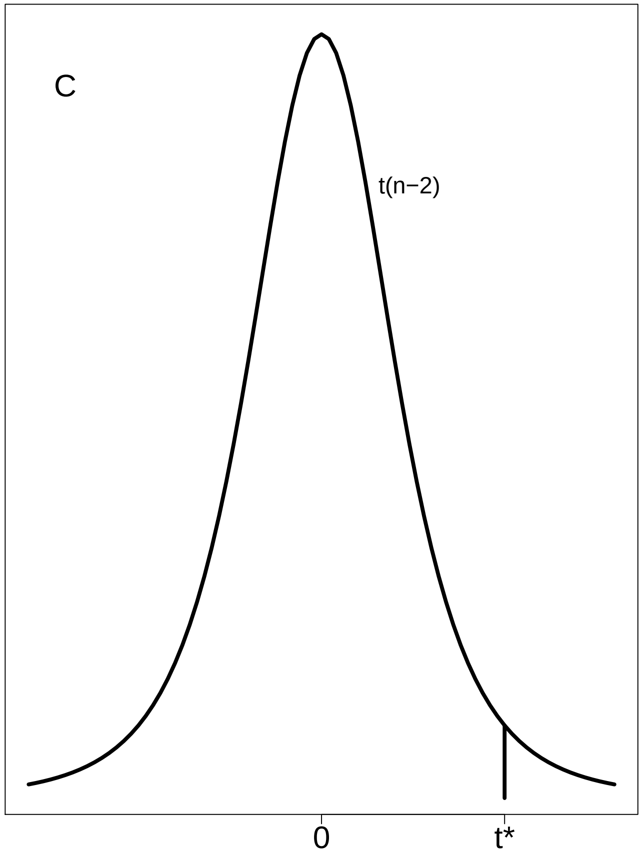
**Rejection Rules:**

**A**: reject $H_0$ if $|t^*| > t(1 - \alpha/2; n - 2)$
**B**: reject $H_0$ if $t^* \leq -t(1 - \alpha; n - 2)$
**C**: reject $H_0$ if $t^* > t(1 - \alpha; n - 2)$

**P-Value:** This is the probability of a $\mathit{more\ extreme}\ t^*$ value than the one we got, given that $H_0$ is true.




B
t(n−2)
t*
0


C
t(n−2)
0
t*

**Example of how to do Hypothesis Tests:**

**Question:** Test $H_0 : \beta_1 = 0$ vs. $H_A : \beta_1 \neq 0$ at level $\alpha = 0.05$ for the house prices data. What is the p-value?

$b_1 = 75.6$, $S_{XX} = 25.38$, $\text{MSE} = 379.21$

If $H_0$ is true, then there is no linear relationship between $\text{E}(Y)$ and square footage.

**Answer:** $H_0 : \beta_1 = 0$, $H_A : \beta_1 \neq 0$, $\alpha = 0.05$

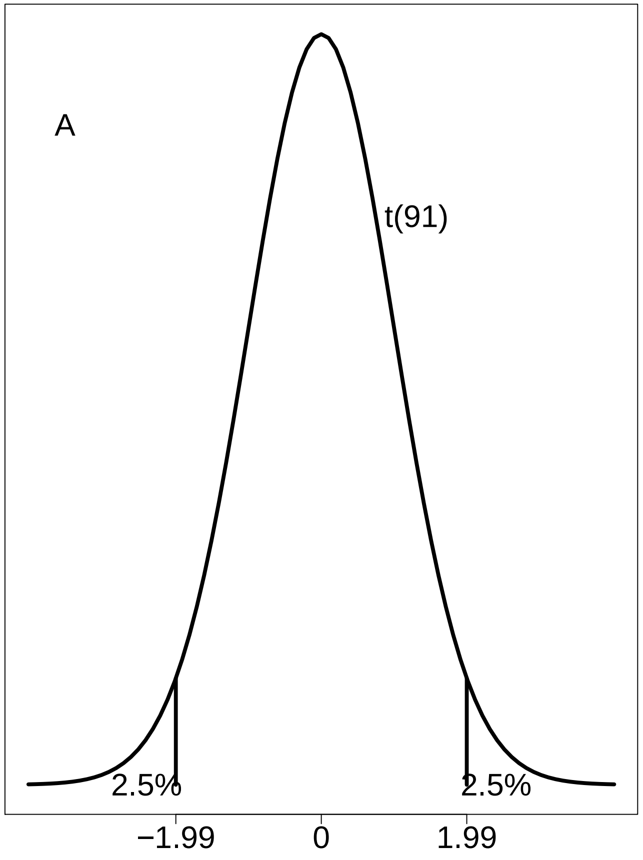
**Test Statistic:**

$$t^* = \frac{b_1 - 0}{\sqrt{\text{MSE}/S_{XX}}} = \frac{75.6}{\sqrt{379.21/25.38}} = 19.56$$

**Rejection Rule:** Reject $H_0$ if $|t^*| > t(1 - \alpha/2; n - 2) = t(0.975; 91) = 1.99$.

**Conclusion:** Reject $H_0$ since $19.56 = |t^*| > t(0.975; 91) = 1.99$. There is a significant linear relationship between mean house price and square footage.

## Example cont'ed: What's the picture?



**Reconsider rejection rule:**

$$\begin{aligned} P(\text{reject } H_0 | H_0 \text{ true}) &= P(|t^*| > 1.99 | H_0 \text{ true}) \\ &= 1 - 0.95 = \alpha \end{aligned}$$

Where is $t^*$ on this picture?

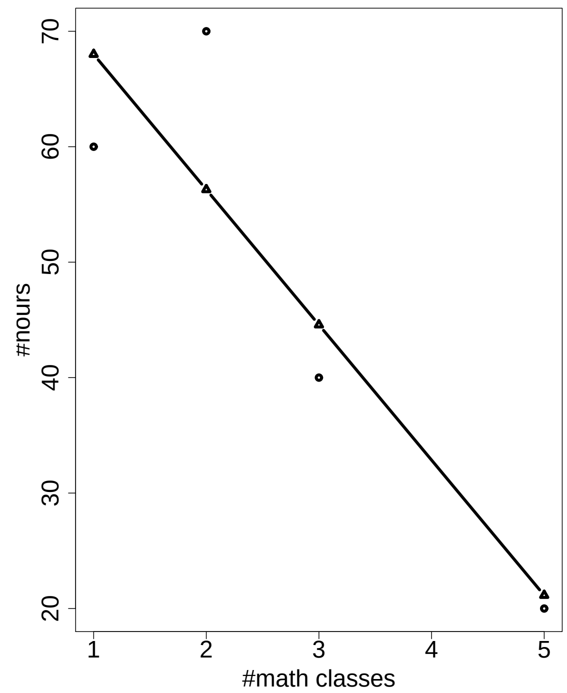
I would have rejected $H_0$ **for any** $|t^*| > 1.99$ !

**P-Value:** Prob of a more extreme $t^*$ is almost 0.

## Extrapolation is Bad!

Never use estimated regression function $\widehat{\mathsf{E}}(Y) = b_0 + b_1 X$ outside the range of $X$ values in the data!

Remember the math class/hours on papers example




My friend is taking 7 math classes next semester. How many hours will he spend writing papers?

$80 - 11.7(7) = -1.9 \quad \Rightarrow$ Nice concept, but wrong!

## Confidence Intervals for Mean Response

Let $X_h$ denote the level of $X$ for which we wish to estimate the mean response $\mathsf{E}(Y_h) = \beta_0 + \beta_1 X_h$.
$X_h$ may be a value which occurred in the sample, or some other value within the scope of the model.

Point estimator $\hat{Y}_h$ of $\mathsf{E}(Y_h)$ is

$$\hat{Y}_h = b_0 + b_1 X_h$$

Notify that with $b_0 = \sum_i l_i Y_i$ and $b_1 = \sum_i k_i Y_i$ we get

$$\hat{Y}_h = \sum_{i=1}^{n} l_i Y_i + X_h \sum_{i=1}^{n} k_i Y_i = \sum_{i=1}^{n} \Big( l_i + X_h k_i \Big) Y_i$$

Thus $\hat{Y}_h$ is normally dist'd and we can figure out its mean and variance:

$$\begin{aligned} \mathsf{E}(\hat{Y}_h) &= \beta_0 + \beta_1 X_h \\ \mathsf{var}(\hat{Y}_h) &= \sigma^2 \left\{ \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right\} \end{aligned}$$

Together we have

$$\hat{Y}_h \sim N\left(\beta_0 + \beta_1 X_h, \sigma^2 \left\{ \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right\}\right)$$

or

$$\frac{\hat{Y}_h - (\beta_0 + \beta_1 X_h)}{\sqrt{\sigma^2 \left\{ \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right\}}} \sim N(0, 1)$$

Plug in MSE for the unknown $\sigma^2$ gives

$$\frac{\hat{Y}_h - (\beta_0 + \beta_1 X_h)}{\sqrt{\mathsf{MSE}\left\{\frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}}\right\}}} \sim t(n-2)$$

Just like for $\beta_1$, a $(1-\alpha)100\%$ CI for $\beta_0 + \beta_1 X_h$ is

$$\hat{Y}_h \pm t(1-\alpha/2; n-2)\sqrt{\mathsf{MSE}\left\{\frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}}\right\}}$$

**Example:** Recall for the house data

$$\widehat{\mathsf{E}}(price) = -25.2 + 75.6(area)$$

$S_{XX} = 25.38$, MSE $= 379.21$, $\bar{X} = 1.65$

Suppose you are thinking of constructing several 2,000 sq.ft. homes in G'ville and you want to know about how much these houses will sell for.

Point estimate is $\widehat{\mathsf{E}}(price) = -25.2 + 75.6(2) = 126$

A 95% CI for $\beta_0 + \beta_1(2)$ is

$$126 \pm t(0.975; 91)\sqrt{379.21\left\{\frac{1}{93} + \frac{(2-1.65)^2}{25.38}\right\}} = 126 \pm 4.86 \approx (121, 131).$$

Thus, we are 95% confident that the mean selling price of 2,000 sq.ft. houses is between 121,000\$ and 131,000\$. (CI for $\mathsf{E}(Y_h)$ is smallest for $X_h = \bar{X}$)

# Prediction Interval for $Y_{h(new)}$

After we collect the data, we might be interested in predicting a new observation whose $X$ value is $X_h$.

Before, we estimated the mean of the distribution of $Y$. Now we predict an individual outcome drawn from the distribution of $Y$.

**Example:** There is a 2,000 sq.ft. house about to be put up for sale. Its price is a r.v. $Y_{h(new)}$ and $X_h = 2$.

Suppose that $\beta_0$ and $\beta_1$ are known.

Question: What do we expect $Y_{h(new)}$ to be?
Answer: $Y_{h(new)} = \beta_0 + \beta_1 X_h + \epsilon_{h(new)}$

So E$(Y_{h(new)}) = \beta_0 + \beta_1 X_h$, var$(Y_{h(new)}) = \sigma^2$ and

$$Y_{h(new)} \sim N(\beta_0 + \beta_1 X_h, \sigma^2)$$

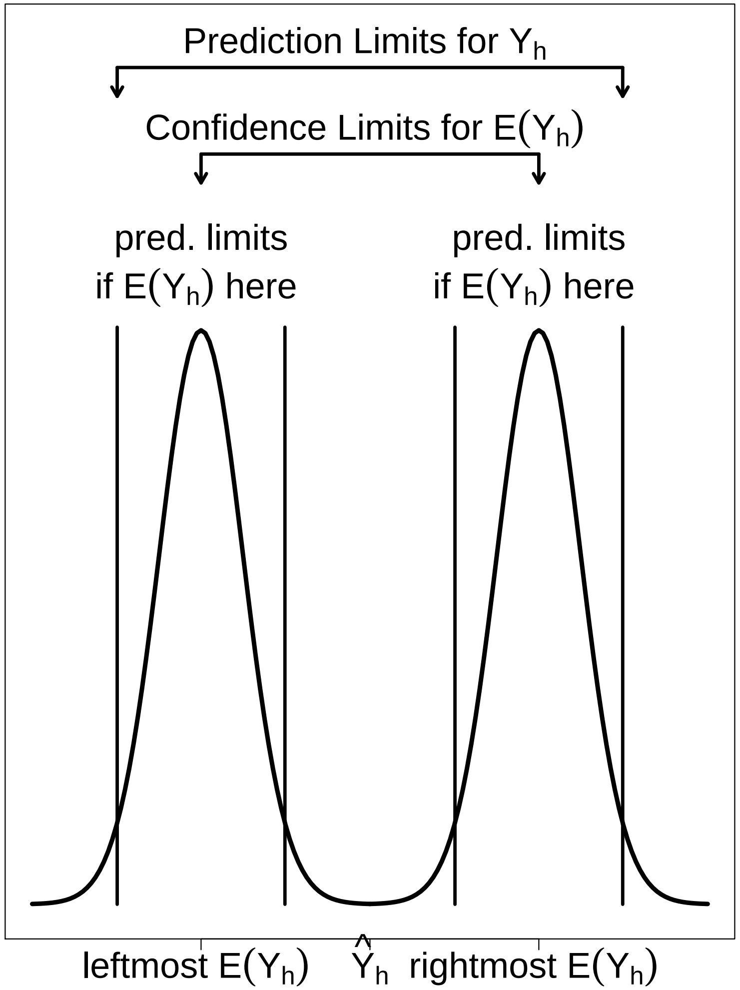
Thus the $1-\alpha$ prediction limits for $Y_{h(new)}$ are:

$$\mathsf{E}(Y_{h(new)}) \pm z(1-\alpha/2)\sigma.$$

Anyway, we don't know the parameters. But we have a $(1-\alpha) * 100\%$ CI for $\beta_0 + \beta_1 X_h$:

$$(b_0 + b_1 X_h) \pm t(1-\alpha/2; n-2)\sqrt{\mathsf{MSE}\left\{\frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}}\right\}}$$

Dist'ns of $Y_{h(new)}$ at the upper and lower CI limit.


Prediction Limits for $Y_h$
Confidence Limits for $E(Y_h)$
pred. limits
if $E(Y_h)$ here
pred. limits
if $E(Y_h)$ here
leftmost $E(Y_h)$
$\hat{Y}_h$
rightmost $E(Y_h)$

The $(1-\alpha)*100\%$ Prediction Interval for $Y_{h(new)}$ is slightly wider than the $(1-\alpha)*100\%$ CI for $\beta_0+\beta_1 X_h$.

We consider the difference

$$Y_{h(new)} - \hat{Y}_h = Y_{h(new)} - \sum_{i=1}^{n}(l_i + X_h k_i)Y_i$$

where $\hat{Y}_h = b_0 + b_1 X_h$ is indep. of $Y_{h(new)}$. Because it's a linear combination, it's a normal variate with

$$\mathsf{E}(Y_{h(new)} - \hat{Y}_h) = \mathsf{E}(Y_{h(new)}) - \mathsf{E}(\hat{Y}_h) = 0$$

and

$$
\begin{aligned}
\text{var}(Y_{h(new)} - \hat{Y}_h) &= \text{var}(Y_{h(new)}) + \text{var}(\hat{Y}_h) \\
&= \sigma^2 + \sigma^2 \left\{ \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right\} \\
&= \sigma^2 \left\{ 1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right\}
\end{aligned}
$$

Thus $(Y_{h(new)} - \hat{Y}_h)/\sqrt{\text{var}(Y_{h(new)} - \hat{Y}_h)} \sim N(0,1)$

$$
\frac{Y_{h(new)} - \hat{Y}_h}{\sqrt{\text{MSE}\left\{1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}}\right\}}} \sim t(n-2)
$$

and a $(1-\alpha)*100\%$ PI for $Y_{h(new)}$ is given by:

$$\hat{Y}_h \pm t(1-\alpha/2; n-2)\sqrt{\text{MSE}\left\{1+\frac{1}{n}+\frac{(X_h-\bar{X})^2}{S_{XX}}\right\}}$$

**Example:** A 95% Prediction Interval for $Y_{h(new)}$, the price of the 2,000 sq.ft. house is

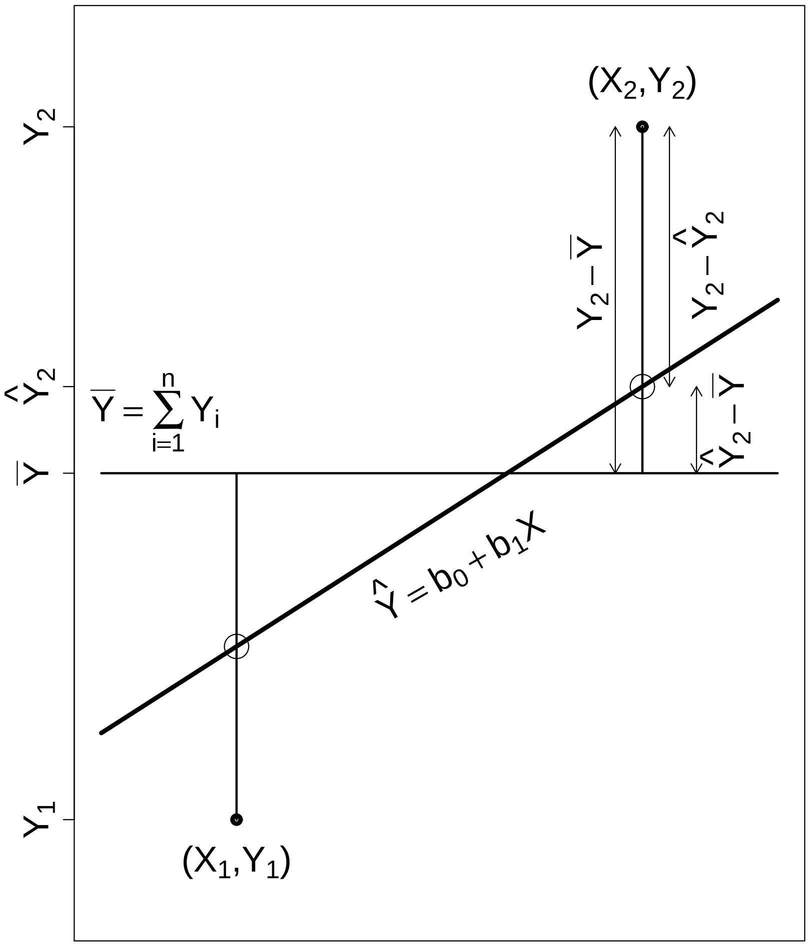
$$126 \pm t(0.975; 91)\sqrt{379.21\left\{1+\frac{1}{93}+\frac{(2-1.65)^2}{25.38}\right\}} = 126 \pm 38.5 \approx (87.5, 164.5).$$

Thus, there is a 95% probability that the price of the house will be between 87,500$ and 164,500$.

# ANalysis Of Variance: ANOVA

**Nothing new**, just a different way of looking at what we have already done.

Say we have the LS estimates of $\beta_0$, $\beta_1$



$(X_2, Y_2)$
$Y_2$
$Y_2 - \bar{Y}$
$Y_2 - \hat{Y}_2$
$\hat{Y}_2$
$\bar{Y} = \sum_{i=1}^{n} Y_i$
$\hat{Y}_2 - \bar{Y}$
$\bar{Y}$
$\hat{Y} = b_0 + b_1 X$
$Y_1$
$(X_1, Y_1)$

Consider the linear relationship $(Y_i - \bar{Y}) = (\hat{Y}_i - \bar{Y}) + (Y_i - \hat{Y}_i)$
Is there a quadratic analogue?
*Total Sum of Squares*: the variation in the $Y$'s if we forget about $X$

$$SSTO = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

*Regression Sum of Squares*: the variation in $Y$'s explained at $X$

$$SSR = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

*Error Sum of Squares*: the variation in $Y$'s around the regression line

$$\mathsf{SSE} = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

Does the partition SSTO $=$ SSR $+$ SSE hold? **Yes!**

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

Generally, in ANOVA methods, the $SSTO$ is partitioned into several sums of squares which each have an associated **degrees of freedom (df)**.

**ANOVA Table for SLR:**

| Source variat. | Sum of Squares (SS) | df | mean SS |
|---|---|---|---|
| Regr. | SSR $= \sum_i (\hat{Y}_i - \bar{Y})^2$ | 1 | $\frac{\text{SSR}}{1}$ |
| Error | SSE $= \sum_i (Y_i - \hat{Y}_i)^2$ | $n-2$ | $\frac{\text{SSE}}{n-2}$ |
| Total | SSTO $= \sum_i (Y_i - \bar{Y})^2$ | $n-1$ | |

Another way to test $H_0 : \beta_1 = 0$ vs. $H_A : \beta_1 \neq 0$

Test statistic:

$$F^* = \frac{\mathsf{MSR}}{\mathsf{MSE}}$$

Rejection rule: reject $H_0$ if $F^* > F(1-\alpha; 1, n-2)$

Fact: F-test and t-test are equivalent; that is the F-test rejects if and only if the t-test rejects.

**Notice:** using $b_0 = \bar{Y} - b_1 \bar{X}$ results in

$$\begin{aligned} SSR &= \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = \sum_{i=1}^{n} (b_0 + b_1 X_i - \bar{Y})^2 = \sum_{i=1}^{n} (\bar{Y} - b_1 \bar{X} + b_1 X_i - \bar{Y})^2 \\ &= b_1^2 \sum_{i=1}^{n} (-\bar{X} + X_i)^2 = b_1^2 S_{XX} \end{aligned}$$

Thus

$$F^* = \frac{b_1^2 S_{XX}}{\text{MSE}} = \frac{b_1^2}{\text{MSE}/S_{XX}} = \left( \frac{b_1}{\sqrt{\text{MSE}/S_{XX}}} \right)^2 = (t^*)^2$$

Generally, if $T \sim t(n-2)$ then $T^2 \sim F(1, n-2)$

## Coefficient of Determination, $r^2$

**Question:** How strong is the **linear** relationship between $Y$ and $X$?

Remember: $\text{SSTO} = \text{SSR} + \text{SSE}$

Define:

$$r^2 = \frac{\text{SSR}}{\text{SSTO}} = \frac{\text{SSTO} - \text{SSE}}{\text{SSTO}} = 1 - \frac{\text{SSE}}{\text{SSTO}} \qquad \text{with } 0 \leq r^2 \leq 1.$$

The higher the $r^2$, the stronger the linear relationship!

Extreme cases:

- $\hat{Y}_i = Y_i$: then $\text{SSE} = 0 \Rightarrow r^2 = 1$
- $b_1 = 0 \Rightarrow \hat{Y}_i = \bar{Y}$: then $\text{SSR} = 0 \Rightarrow r^2 = 0$

**BUT:** $r^2 \approx 0$ does not always mean that there is **no** relationship at all between $Y$ and $X$! It only means that the relationship is **not linear**!