AUSTRIAN JOURNAL OF STATISTICS
Volume 41 (2012), Number 3, 175–196

An Overview of FIGARCH and Related Time Series Models

Maryam Tayefi and T. V. Ramanathan
Department of Statistics and Centre for Advanced Studies
University of Pune, India

Abstract: This paper reviews the theory and applications related to fractionally integrated generalized autoregressive conditional heteroscedastic (FIGARCH) models, mainly for describing the observed persistence in the volatility of a time series. The long memory nature of FIGARCH models allows to be a better candidate than other conditional heteroscedastic models for modeling volatility in exchange rates, option prices, stock market returns and inflation rates. We discuss some of the important properties of FIGARCH models in this review. We also compare the FIGARCH with the autoregressive fractionally integrated moving average (ARFIMA) model. Problems related to parameter estimation and forecasting using a FIGARCH model are presented. The application of a FIGARCH model to exchange rate data is discussed. We briefly introduce some other models, that are closely related to FIGARCH models. The paper ends with some concluding remarks and future directions of research.

Zusammenfassung: Dieser Aufsatz bespricht die Theorie und Anwendungen im Zusammenhang mit *Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic* (FIGARCH) Modellen, vor allem für die Beschreibung der beobachteten Persistenz in der Volatilität einer Zeitreihe. Die *Long Memory* Natur von FIGARCH Modellen ermöglicht es, ein besserer Kandidat als andere bedingte heteroskedastische Modelle zur Modellierung der Volatilität bei Wechselkursen, Optionspreisen, Aktienrenditen und Inflationsraten zu sein. Wir diskutieren einige der wichtigsten Eigenschaften von FIGARCH Modellen in diesem Review. Wir vergleichen auch das FIGARCH Modell mit dem *Autoregressive Fractionally Integrated Moving Average* (ARFIMA) Modell. Probleme im Zusammenhang mit der Parameterschätzung und der Prognose mit FIGARCH Modellen werden vorgestellt. Die Anwendung eines FIGARCH Modells auf Wechselkursdaten wird diskutiert. Kurz werden einige andere Modelle vorgestellt, die eng mit FIGARCH Modellen verwandt sind. Der Beitrag endet mit abschließenden Bemerkungen und zukünftige Ausrichtung der Forschung.

Keywords: ARCH, ARFIMA, FIGARCH, GARCH, Long Memory Models, Volatility Models.

1 Introduction

Volatility is a term that has been extensively used in financial applications. It refers to the conditional standard deviation of the underlying asset return. Volatility modelling provides a simple approach of calculating value-at-risk of a financial position in risk management. Furthermore, modelling the volatility of a time series can improve the efficiency in parameter estimation and the accuracy of forecast. In time series literature, models which attempt to explain the changes in conditional variance are generally known as conditional heteroscedastic models. Some of the volatility models that have been extensively used in the literature are Autoregressive Conditional heteroscedastic (ARCH) model of Engle (1982), Generalized Autoregressive Conditional Heteroscedastic (GARCH) model of Bollerslev (1986), Integrated GARCH (IGARCH) model of Engle and Bollerslev (1986) and Fractionally Integrated GARCH (FIGARCH) model of Baillie, Bollerslev, and Mikkelsen (1996).

Under the ARCH framework, it is generally assumed that large shocks tend to follow large shocks and similarly, the small shocks tend to follow small shocks, a phenomena known as volatility clustering. Although the ARCH model is simple, it often requires many parameters to adequately describe the volatility process of an asset return. Therefore, some alternative models were introduced. Bollerslev (1986) proposed a very useful extension of ARCH model, known as GARCH. The GARCH model is simply an infinite order ARCH with exponentially decaying weights for distant lags.

If the AR polynomial of the GARCH representation has a unit root, then we have an Integrated GARCH model (IGARCH), which was first introduced by Engle and Bollerslev (1986). A key feature of IGARCH model is that the impact of the past squared shock is persistent and the pricing of risky securities, including long-term options and future contracts, may show extreme dependence on the initial conditions. Several studies report the presence of apparent long-memory in the autocorrelations of squared or absolute returns of various financial assets. Motivated by these observations, Baillie et al. (1996) introduced the Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) process.

The primary purpose of introducing FIGARCH model was to develop a more flexible class of processes for the conditional variance, that are capable of explaining and representing the observed temporal dependencies in financial market volatility. In particular, the FIGARCH model allows only a slow hyperbolic rate of decay for the lagged squared or absolute innovations in the conditional variance function. This model can accommodate the time dependence of the variance and a leptokurtic unconditional distribution for the returns with a long memory behaviour for the conditional variances.

This paper has been framed in such a way that, in Section 2, we review the volatility models that led to FIGARCH. We also introduce here the FIGARCH model and discuss some of its properties. In Section 3, parameter estimation procedures such as MLE, QMLE for the FIGARCH model will be explained. Also in this section we briefly discuss forecasting with a FIGARCH model. Applications of FIGARCH model constitute Section 4. In Section 5, we consider models which are very closely related to FIGARCH, such as, adaptive-FIGARCH (A-FIGARCH), hyperbolic GARCH (HYGARCH), smooth transition FIGARCH (ST-FIFARCH) and Asymmetric FIGARCH and explain how they cover up some of the limitations of FIGARCH. The paper ends with some concluding remarks and future research directions in Section 6.

2 Conditional Heteroscedastic Models and FIGARCH

2.1 Conditional Heteroscedastic Models

Let ϵ_t denote a real-valued discrete-time stochastic process, and ψ_t be the information set of all information up to time t, i.e., $\psi_t = \sigma\{\ldots, \epsilon_{t-2}, \epsilon_{t-1}, \epsilon_t\}$. The process $\{\epsilon_t\}$ is said to be an ARCH(q), whenever

$$\mathrm{E}(\epsilon_t|\psi_{t-1}) = 0 \qquad \text{and} \qquad \mathrm{var}(\epsilon_t|\psi_{t-1}) = h_t\,, \tag{1}$$

with

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2\,,$$

where $\alpha_i \geq 0$, $i = 1, \ldots, q$. The conditional variance can be generally expressed as

$$h_t = h(\epsilon_{t-1}, \epsilon_{t-2}, \ldots, \epsilon_{t-q}, \boldsymbol{\alpha})\,,$$

where $h(\cdot)$ is a nonnegative function of its arguments and $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_q)'$ is a vector of unknown parameters. The expression (1) is sometimes represented as

$$\epsilon_t = h_t^{1/2} z_t \tag{2}$$

with $\mathrm{E}(z_t) = 0$, $\mathrm{var}(z_t) = 1$, where the z_t's are uncorrelated. It may be noted that the $\{\epsilon_t\}$ process is serially uncorrelated with mean zero and the conditional variance h_t, which is changing over time.

Lots of research has been done to address various problems related to ARCH models after its introduction by Engle (1982). Xekalaki and Degiannakis (2010) provide some of the recent developments related to ARCH models.

In empirical applications of the ARCH model, a relatively long lag in the conditional variance equation is often called for and to avoid problems with negative variance parameter estimates, a fixed lag structure is typically imposed on. Because of these reasons, there is a practical interest to extend the ARCH class of models to permit for both a longer memory and a more flexible lag structure.

The GARCH(p, q) process introduced by Bollerslev (1986) and Taylor (1986) (independently of each other) is given by (1) along with the volatility equation

$$\begin{aligned} h_t &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \\ &= \alpha_0 + \alpha(L)\epsilon_t^2 + \beta(L) h_t\,, \end{aligned} \tag{3}$$

where $p > 0$, $q > 0$, $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, q$, $\beta_j \geq 0$, $j = 1, \ldots, p$ and $\alpha(L)$ and $\beta(L)$ are lag operators such that $\alpha(L) = \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_q L^q$ and $\beta(L) = \beta_1 L + \beta_2 L^2 + \cdots + \beta_p L^p$. For $p = 0$, the process reduces to an ARCH(q) and for $p = q = 0$, ϵ_t is simply a white noise process.

The GARCH(p, q) process as defined in (3) is wide stationary with $\mathrm{E}(\epsilon_t) = 0$ and $\mathrm{var}(\epsilon_t) = \alpha_0(1 - \alpha(1) - \beta(1))^{-1}$. An equivalent ARMA type representation of the GARCH(p, q) process is given by

$$\epsilon_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \epsilon_{t-j}^2 - \sum_{j=1}^{p} \beta_j \nu_{t-j} + \nu_t \,, \tag{4}$$

where $\nu_t = \epsilon_t^2 - h_t = (z_t^2 - 1)h_t$ and the z_t's are uncorrelated with $\mathrm{E}(z_t) = 0$ and $\mathrm{var}(z_t) = 1$.

Huge amount of empirical and theoretical research work has been already done for GARCH and related models. We refer to part 1 of Andersen, Davis, Kreis, and Mikosch (2009) and Francq and Zakoian (2010), which deal with almost all developments related to GARCH models. Rohan (2009) had given an excellent review on asymmetric GARCH models. Also see Rapach and Strauss (2008), Smith (2008), Rohan and Ramanathan (2012) and C. S. Li and Xiao (2011) for some of the recent developments on GARCH models with structural breaks.

Engle and Bollerslev (1986) considered a particular class of GARCH models known as integrated GARCH (IGARCH) models whose unconditional variance does not exist. This occurs when $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j = 1$ in a GARCH(p, q) model. Similar to ARIMA models, a key feature of IGARCH models is that the impact of past squared shocks $\nu_{t-i} = \epsilon_{t-i}^2 - h_{t-i}$ for $i > 0$ on ϵ_t^2 is persistent. An IGARCH(1, 1) model can be written as:

$$\epsilon_t = z_t h_t^{1/2} \,, \qquad h_t = \alpha_0 + \beta_1 h_{t-1} + (1 - \beta_1)\epsilon_{t-1}^2 \,,$$

where $\{z_t\}$ is defined as before and $0 < \beta_1 < 1$. However, it is interesting to note that the IGARCH model can be strongly stationary even though it is not weakly stationary. The IGARCH model implies infinite persistence of the conditional variance to a shock in squared returns. On the other hand in most of the empirical situations the volatility process is found to be mean reverting. Thus the IGARCH model seems to be too restrictive as it implies infinite persistence of a volatility shock.

2.2 FIGARCH Process

From (4) we see that a GARCH(p, q) process may also be expressed as an ARMA(m, p) process in ϵ_t^2, by writing

$$[1 - \alpha(L) - \beta(L)]\epsilon_t^2 = \alpha_0 + [1 - \beta(L)]\nu_t \,,$$

where $m = \max\{p, q\}$ and $\nu_t = \epsilon_t^2 - h_t$. The $\{\nu_t\}$ process can be interpreted as the "innovations" for the conditional variance, as it is a zero-mean martingale. Therefore, an integrated GARCH(p, q) process can be written as

$$[1 - \alpha(L) - \beta(L)](1 - L)\epsilon_t^2 = \alpha_0 + [1 - \beta(L)]\nu_t \,. \tag{5}$$

The fractionally integrated GARCH or FIGARCH class of models is obtained by replacing the first difference operator $(1 - L)$ in (5) with the fractional differencing operator

$(1-L)^d$, where d is a fraction $0 < d < 1$. Thus, the FIGARCH class of models can be obtained by considering

$$[1-\alpha(L)-\beta(L)](1-L)^d\epsilon_t^2 = \alpha_0 + [1-\beta(L)]\nu_t\,.$$

Such an approach can develop a more flexible class of processes for the conditional variance that are capable of explaining and representing the observed temporal dependencies of the financial market volatility in a much better way than other types of GARCH models (Davidson, 2004).

It may be noted that the fractional differencing operator $(1-L)^d$ can be written in terms of hypergeometric function,

$$(1-L)^d = F(-d,1,1;L) = \sum_{k=0}^{\infty}\Gamma(k-d)\Gamma(k+1)^{-1}\Gamma(-d)^{-1}L^k\,. \tag{6}$$

The ARFIMA(p,d,q) class of models for the discrete time real-valued process $\{y_t\}$ introduced by Granger and Joyeux (1980); Granger (1980, 1981) and Hosking (1981) is defined by

$$a(L)(1-L)^d y_t = b(L)z_t\,, \tag{7}$$

where $a(L)$ and $b(L)$ are polynomials in the lag operator of orders p and q respectively, and $\{z_t\}$ is a mean-zero serially uncorrelated process. For the ARFIMA models, the fractional parameter d lies between $-1/2$ and 1/2, (Hosking, 1981). The ARFIMA model is nothing but the fractionally integrated ARMA for the mean process. Analogous to the ARFIMA(p,d,q) process defined in (7) for the mean, the FIGARCH(p,d,q) process for $\{\epsilon_t^2\}$ can be defined as

$$\phi(L)(1-L)^d\epsilon_t^2 = \alpha_0 + [1-\beta(L)]\nu_t\,, \tag{8}$$

where $0 < d < 1$, and all the roots of $\phi(L)$ and $[1-\beta(L)]$ lie outside the unit circle. In the case of ARFIMA model, the long memory operator is applied to unconditional mean μ of y_t which is constant. But this is not true in the case of FIGARCH model, where it is not applied to α_0, but on squared errors.

Bordignon, Caporin, and Lisi (2004) have introduced a FIGARCH model with seasonlity, which allows for both periodic patterns and long memory behaviour in the conditional variance. It can also merge these two aspects allowing the model to be both periodic and having long memory components. Such a model is given by

$$h_t = \alpha_0 + \alpha(L)\epsilon_t^2 + \beta(L)h_t + [1-(1-L^S)^d]\epsilon_t^2\,,$$

the first three terms in the conditional variance reproduce the general GARCH model, the fourth term introduces a long memory component which operates at zero and seasonal frequencies. The parameter S represents the length of the cycle, while d indicates the degree of memory.

Rearranging the terms in (8), an alternative representation for the FIGARCH(p,d,q) model may be obtained as

$$[1-\beta(L)]h_t = \alpha_0 + [1-\beta(L)-\phi(L)(1-L)^d]\epsilon_t^2\,. \tag{9}$$

From (9), the conditional variance h_t of y_t is given by

$$\begin{aligned} h_t &= \alpha_0[1-\beta(1)]^{-1} + \left\{1 - [1-\beta(L)]^{-1}\phi(L)(1-L)^d\right\}\epsilon_t^2 \\ &= \alpha_0[1-\beta(1)]^{-1} + \lambda(L)\epsilon_t^2\,, \end{aligned} \tag{10}$$

where $\lambda(L) = \lambda_1 L + \lambda_2 L^2 + \dots$. Of course, for the FIGARCH(p,d,q), for (8) to be well-defined, the conditional variance in the ARCH(∞) representation in (10) must be non-negative, i.e., $\lambda_k \geq 0$ for $k = 1, 2, \dots$. Some sufficient conditions are available in the literature. In the next subsection, we discuss them in detail.

2.2.1 Nonnegativity of Conditional Variance

A sufficient condition for the nonnegativity of the conditional variance of the FIGARCH $(1,d,1)$ model is available from the literature. This condition was suggested by Baillie et al. (1996) and by Bollerslev and Mikkelsen (1996) using the nonnegativity of the λ_k coefficients of the FIGARCH$(1,d,1)$ model. In FIGARCH$(1,d,1)$,

$$h_t = \alpha_0(1-\beta(1))^{-1} + [1 - (1-\beta_1 L)^{-1}(1-\phi_1 L)(1-L)^d]\epsilon_t^2\,,$$

where

$$\lambda(L) = \lambda_1 L + \lambda_2^2 L + \dots = 1 - [(1-\beta_1 L)^{-1}(1-\phi_1 L)(1-L)^d]\,.$$

Therefore, equating the coefficients

$$\begin{aligned} \lambda_1 &= \phi_1 - \beta_1 + d \\ \lambda_2 &= (d-\beta_1)(\beta_1-\phi_1) + \frac{d(1-d)}{2} \\ \lambda_3 &= \beta_1\left[d\beta_1 - d\phi_1 - \beta_1^2 + \beta_1\phi_1 + \frac{d(1-d)}{2}\right] + d\frac{1-d}{2}\left(\frac{2-d}{3} - \phi_1\right) \\ &\vdots \\ \lambda_k &= \beta_1\lambda_{k-1} + \left(\frac{k-1-d}{k} - \phi_1\right)\delta_{d,k-1}\,, \qquad k = 2, 3, \dots\,, \end{aligned}$$

where $\delta_{d,k} = \delta_{d,k-1}(k-1-d)k^{-1}$ refer to the coefficients in the series expansion of $(1-L)^d$ for $k = 2, 3, \dots$. That is,

$$\delta_d(L) = \sum_{k=1}^{\infty} \delta_{d,k} L^k\,,$$

with $\delta_{d,0} = 1$. Using the non-negativity of λ_k's, it is possible to derive inequalities which are sufficient for all conditional variances h_t to be non-negative:

$$\beta_1 - d \leq \phi_1 \leq \frac{2-d}{3} \qquad \text{and} \qquad d\left(\phi_1 - \frac{1-d}{2}\right) \leq \beta_1(d - \beta_1 + \phi_1)\,.$$

Restrictions for lower order models can be derived similarly, while for higher order models such restrictions for the parameters cannot be derived easily (Caporin, 2003).

When we estimate the parameters of FIGARCH model, occasionally it fails to satisfy the Bollerslev and Mikkelsen (1996) condition that all parameters should be positive to ensure the nonnegativity of conditional variance. This would lead any researcher to reject the model according to these conditions. For this reason, Conrad and Haag (2006) introduced another set of conditions that guarantees the nonnegativity of the conditional variance in all situations.

Theorem 1: (Conrad and Haag, 2006). *Let the coefficients g_j and f_i be the functions of the fractional differencing parameter* d *such that* $g_j = f_j \cdot g_{j-1} = \prod_{i=1}^{j} f_i$ *with* $f_j = \frac{j-1-d}{j}$ *for* $j = 1, 2, \ldots$ *and* $g_0 = 1$. *Then the conditions are*
$\psi_1 = d + \phi_1 - \beta_1$ *and* $\psi_i = \beta_1 \psi_{i-1} + (f_i - \phi_1)(-g_{i-1})$, $i \geq 2$, *or alternatively,*
$\psi_i = \beta_1^2 \psi_{i-2} + [\beta_1(f_{i-1}\phi_1) + (f_i - \phi_1) f_{i-1}](-g_{i-2})$ *for all* $i \geq 3$.

2.2.2 Impulse Response Function of FIGARCH

When the conditional variance is parametrized as a linear function of the past squared innovations, the persistence of the conditional variance is simply characterized in terms of the impulse response coefficients defined by

$$\begin{aligned}\gamma_k &= \frac{\partial \mathrm{E}(\epsilon_{t+k}^2 | \psi_t)}{\partial \nu_t} - \frac{\partial \mathrm{E}(\epsilon_{t+k-1}^2 | \psi_t)}{\partial \nu_t} \\ &= \frac{\partial h_{t+k}}{\partial \nu_t} - \frac{\partial h_{t+k-1}}{\partial \nu_t},\end{aligned}$$

where $\nu_t = \epsilon_t^2 - h_t$. From the ARCH($\infty$) representation of FIGARCH given in (10),

$$\begin{aligned}\frac{\partial h_{t+k}}{\partial \nu_t} &= \lambda(L) \frac{\partial \epsilon_{t+k}^2}{\partial \nu_t} \\ &= \lambda(L) \tfrac{\partial}{\partial \nu_t} \{\alpha_0 \phi(L)^{-1}(1-L)^{-d} + \phi(L)^{-1}(1-L)^{-d}[1-\beta(L)]\nu_{t+k}\} \\ &= \lambda(L)\phi(L)^{-1}(1-L)^{-d}(-\beta_k) \\ &= -\beta_k\{1 - [1-\beta(L)]^{-1}\phi(L)(1-L)^d\}\phi(L)^{-1}(1-L)^{-d} \\ &= -\beta_k \phi(L)^{-1}(1-L)^{-d} + [1-\beta(L)]^{-1}.\end{aligned}$$

Similarly,

$$\frac{\partial h_{t+k-1}}{\partial \nu_t} = -\beta_{k-1}\phi(L)^{-1}(1-L)^{-d} + [1-\beta(L)]^{-1}.$$

Therefore,

$$\gamma_k = (\beta_{k-1} - \beta_k)\phi(L)^{-1}(1-L)^{-d}.$$

Generally for conditional variance models, the γ_i's will depend on the time t information set. However, for the FIGARCH class of models studied here, the impulse response coefficients are independent of t, which indicate the persistence of conditional variance (Baillie et al., 1996).

Analogous to the conventional impulse response analysis for the mean, the long-run impact of past shocks for the volatility process may be assessed in terms of the limit of the cumulative impulse response weights; i.e.,

$$\begin{aligned}\gamma(1) &= F(d-1,1,1;1)\phi(1)^{-1}[1-\beta(1)] \\ &= \gamma_0 + \gamma_1 + \dots \\ &= \lim_{k\to\infty}\sum_{i=0}^{k}\gamma_i\,,\end{aligned}$$

where F is the hypergeometric function defined in (6). Since, $F(d-1,1,1,L) = (1-L)^{1-d}$, for $0 \leq d < 1$, we have $F(d-1,1,1;1) = 0$, so that for the covariance stationary GARCH(p,q) model and the FIGARCH(p,d,q) model with $0 < d < 1$, shocks to the conditional variance will ultimately die out. For $d = 1$, $F(d-1,1,1;1) = 1$, the cumulative impulse response weights will converge to the nonzero constant $\gamma(1) = \phi(1)^{-1}[1-\beta(1)]$. Thus, from a forecasting perspective, shocks to the conditional variance of the IGARCH model persist indefinitely. For $d > 1$, $F(d-1,1,1;1) = \infty$, resulting in an unrealistic explosive conditional variance process and $\gamma(1)$ being undefined.

By analogy to the properties for the ARFIMA$(0,d,1)$ model, it is possible to find the cumulative impulse response coefficients in ARCH(∞) representation (10) for the FIGARCH$(1,d,0)$ model. The coefficients λ_k may be derived from

$$\lambda(L) = 1 - (1-\beta_1 L)^{-1}(1-L)^d\,.$$

This can be obtained as follows. Let $\{y_t\}$ be a stationary invertible ARFIMA$(1,d,0)$ process

$$(1-\phi L)(1-L)^d y_t = u_t\,.$$

Then the infinite autoregressive representation of $\{y_t\}$ is

$$\sum_{k=0}^{\infty}\pi_k y_{t-k} = u_t\,,$$

where the π_k's can be derived as follows:

$$\begin{aligned}\sum_{k=0}^{\infty}\pi_k y_{t-k} &= (1-\phi L)(1-L)^d y_t \\ &= (1-\phi L)(1-dL-1/2d(1-d)L^2-1/6d(1-d)(2-d)L^3-\dots)y_t\,.\end{aligned}$$

Expanding both sides and equating the coefficients of L^k, we get

$$\pi_k = \frac{(k-d-2)!}{(k-1)!(-d-1)!}\{1-\phi-(1+d)/k\} \sim \frac{(1-\phi)}{(-d-1)!}k^{-d-1}\,.$$

Now consider $\{y_t\}$ to be a stationary invertible ARFIMA$(0,d,1)$. Then the infinite moving average representation of this model equals to the infinite autoregressive representation of ARFIMA$(1,-d,0)$. Therefore, for ARFIMA$(0,d,1)$ we have

$$\pi_k = \frac{(k+d-2)!}{(k-1)!(d-1)!}\{1-\phi-(1-d)/k\} \sim \frac{(1-\phi)}{(d-1)!}k^{d-1}\,.$$

The FIGARCH$(1, d, 0)$ is

$$(1 - L)^d \epsilon_t^2 = \alpha_0 + (1 - \beta_1 L)\nu_t \,. \tag{11}$$

In view of the analogy of (11) to ARFIMA$(0, d, 1)$, we have

$$\lambda_k = \frac{(k + d - 2)!}{(k - 1)!(d - 1)!}\{1 - \beta_1 - (1 - d)/k\} \,,$$

which may be written as

$$\lambda_k = \Gamma(k + d - 1)\Gamma(d)^{-1}\Gamma(k)^{-1}\{1 - \beta_1 - (1 - d)k^{-1}\}$$

for $k > 1$, and $\lambda_0 = 1$. Furthermore, it follows by an application of Stirling's formula, for higher lags k,

$$\lambda_k \approx [(1 - \beta_1)\Gamma(d)^{-1}]k^{d-1}.$$

Davidson (2004) had given some insight on the memory properties of the FIGARCH. According to Davidson (2004), the degree of persistence of the FIGARCH model operates in the opposite direction as that of ARFIMA, as the d parameter gets closer to zero, the memory of the process increases. This is due to the inverse relationship between the integration coefficient and the conditional variance. The memory parameter acts directly on the squared errors, not on the h_t, this particular behaviour may also influence the stationarity properties of the process (Davidson, 2004). These observations are strictly related to the impulse response analysis on the effects of a shock on a system driven by a FIGARCH process. In such a system, a shock ν_t at time t, should be interpreted as the difference between the squared mean-residuals ϵ_t^2 at time t and the one-step-ahead forecast of the variance h_t, made at time $t - 1$. That is, $\nu_t = \epsilon_t^2 - h_t$. This shock is exactly the innovation in the ARMA representation of the FIGARCH process

$$\epsilon_t^2 = \alpha_0 + [1 - \beta(L)][(1 - L)^d \phi(L)]^{-1}\nu_t \,.$$

The shock may be also interpreted as an unexpected volatility variation or as the forecast error of the variance. It may be remembered that the squared residuals are proxy for the variance and that the time t variance depends on time $t - 1$ information set and may be viewed as a one-step-ahead forecast.

2.2.3 Stationarity of FIGARCH

Baillie et al. (1996) used the results of Nelson (1990) and Bougerol and Picard (1992) to claim the strict stationarity and ergodicity of the FIGARCH$(1, d, 0)$ model for $0 < d < 1$. They claimed that stationarity could be verified with a dominance type argument between the sequence of coefficients of the ARCH(∞) representations of the FIGARCH$(1, d, 0)$ and of an appropriately chosen IGARCH(1,1). However, the FIGARCH equation has no stationary solutions with $E\epsilon_t^2 < \infty$ (Giraitis, Leipus, and Surgailis, 2009). In general, the question of the existence of a stationary solution for FIGARCH process with infinite variance remains open. On the stationarity issue, the main works are those of Giraitis, Kokoszka, and Leipus (2000), Kazakevicius and Leipus (2002) and Zaffaroni (2004). All the above cited authors tried to find necessary and sufficient conditions for stationarity of the ARCH(∞) processes. Their approaches are not completely equivalent, and not all of them are directly and easily applicable to the FIGARCH case. In a special case, Douc, Roueff, and Soulier (2008) show that a non-zero stationary solution exists when $\{z_t\}$ is a sequence of iid random variables, such that $\mathrm{E}(z_0^2) = 1$ and $P\{|z_0| = 1\} < 1$. In such a case there exists $d^* \in [0, 1)$ such that for all $d \in (d^*, 1)$, the FIGARCH$(0, d, 0)$ equation has a unique causual stationary solution satisfying $\mathrm{E}[|\epsilon_t|^{2r}] < \infty$ for all $r < 1$. However the proof of stationarity in the general case of FIGARCH(p, d, q) is not yet available.

2.2.4 FIGARCH with NIG Distribution

In GARCH processes, it is commonly assumed that the tails of the empirical distributions of financial market returns are thicker than in the normal case. Barndorff-Nielsen (1997), Anderson (2001) and Jensen and Lunde (2001), have suggested the Normal Inverse Gaussian (NIG) distribution for the errors in GARCH and stochastic volatility models. The density function of an NIG distributed variable X is

$$\begin{aligned}\mathrm{NIG}(x; a, b, \mu, \delta) &= \frac{a}{\pi\delta}\exp\left(\sqrt{a^2 - b^2} + b\frac{x-\mu}{\delta}\right) \\ &\quad \times q\left(\frac{x-\mu}{\delta}\right)^{-1} k_1\left(aq\left(\frac{x-\mu}{\delta}\right)\right),\end{aligned}$$

where $q(y) = \sqrt{1 + y^2}$ with $0 \leq |b| \leq a$, $x, \mu \in \mathbb{R}$ and $\delta > 0$ and $k_1(\cdot)$ denotes the modified Bessel function of the third order and index one (Abramowitz and Stegun, 1965). Jensen and Lunde (2001) showed that the NIG distribution for z_t defined in (2) can model better than the Student-t and standard normal distributions not only the tails of the return distributions, but also the center, where it is relatively more peaked than the standard normal distribution. Kilic (2007) introduced a FIGARCH with NIG distribution. To present the FIGARCH-NIG model, let the return series r_t be written as

$$r_t = \mu + \frac{b\sqrt{\gamma}}{a}\sigma_t + z_t h_t^{1/2}, \qquad t = 1, \ldots, T,$$

where the z_t's are uncorrelated with zero-mean and unit variance. Let the density of z_t be NIG distributed, that is,

$$z_t \sim \mathrm{NIG}\left(a, b, \frac{-b\sqrt{\gamma}}{a}, \frac{\gamma^{3/2}}{a}\right), \qquad \gamma = \sqrt{a^2 - b^2}.$$

This way of specifying the NIG distribution implies that the conditional distribution of returns r_t will be NIG as well. That is,

$$r_t|\psi_{t-1} \sim \mathrm{NIG}\left(a, b, \mu, \frac{\gamma^{3/2}}{a}h_t^{1/2}\right),$$

where ψ_{t-1} is the information set, $\psi_{t-1} = \sigma(r_{t-1}, r_{t-2}, \ldots)$. The conditional expectation and variance are given by

$$\mathrm{E}(r_t|\psi_{t-1}) = \mu + h_t^{1/2}\frac{b\sqrt{\gamma}}{a}, \qquad t = 1, \ldots, T$$

and

$$\text{var}(r_t|\psi_{t-1}) = h_t\,.$$

This parametrization of the NIG distribution allows to model the temporal dependence in the conditional variance of the random variable to be given solely by h_t. Let

$$\begin{aligned} u_t &= r_t - \text{E}(r_t|\psi_{t-1}) \\ &= r_t - h_t^{1/2}\frac{b\sqrt{a^2-b^2}}{a} - \mu \end{aligned}$$

be the innovation of the return process. Then, the FIGARCH-NIG model is obtained by

$$\phi(L)(1-L)^d u_t^2 = \alpha_0 + [1-\beta(L)]\nu_t\,,$$

where $\nu_t = u_t^2 - h_t$ and $0 < d < 1$ as mentioned before.

Davidson (2004) had shown that a FIGARCH model possesses more memory than a GARCH or IGARCH model and hence the FIGARCH model with NIG errors can be quite useful in modelling both hyperbolic memory and other salient features of the series.

3 Estimation and Forecasting

The estimation of parameters of FIGARCH model is generally carried out using the maximum likelihood method (which is most efficient) with normality assumption for z_t. But the normality assumption can be questioned with some empirical evidence and therefore the use of quasi-maximum likelihood estimator is preferred. The likelihood of a FIGARCH(p, d, q) process based on the sample $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_T\}$ may be written as

$$\log L(\theta, \epsilon_1, \epsilon_2, \ldots, \epsilon_T) \simeq -0.5T\log(2\pi) - 0.5\sum_{t=1}^{T}[\log(h_t) + \epsilon_t^2 h_t^{-1}]\,, \tag{12}$$

where $\theta' \equiv (\alpha_0, d, \beta_1, \ldots, \beta_p, \phi_1, \ldots, \phi_q)$. The likelihood function is maximized conditional on the start-up values. In particular, we need to fix all the pre-sample values of ϵ_t^2 for $t = 0, -1, -2, \ldots$ in the infinite ARCH representation in (10) at the unconditional sample variance.

For the FIGARCH(p, d, q) model with $d > 0$, the population variance does not exist. In most practical applications with high frequency financial data, the standardized innovations $z_t = h_t^{-1/2}\epsilon_t$ are leptokurtic and not normally distributed through time. In these situations the robust quasi-MLE (QMLE) procedures discussed by Weiss (1986) and Bollerslev and Wooldridge (1992) may give better results while doing inference.

Baillie et al. (1996) have claimed the asymptotic normality of the quasi-maximum likelihood estimator $\hat{\theta}_T$, when $(\epsilon_1, \ldots, \epsilon_T)$ form a sample from FIGARCH$(1, d, 0)$ by extending a similar result available for IGARCH(1,1), using a dominance-type argument. They have used an upper bound for the infinite sequence of coefficients of the ARCH(∞) representation of an IGARCH model. A similar argument was also used in claiming the asymptotic properties of the quasi-maximum likelihood estimator for the FIGARCH. Mikosch and Starica (2003) point out that Baillie's claims were not completely correct.

In fact, it is not possible to bound a hyperbolically decaying sequence (the ARCH(∞) coefficients of any FIGARCH process) by an exponentially decaying sequence, which affects the proof of the claim (see Caporin, 2003). The reported Monte Carlo experiment in Baillie et al. (1996) and Bollerslev and Mikkelsen (1996) is still valid and can be used to conclude the consistency and asymptotic normality of the QMLE estimators empirically. However, the only model they have cosidered was FIGARCH$(1, d, 0)$. The proofs of consistency and asymptotic normality of the QMLE estimator is still not resolved for the general FIGARCH(p, d, q) model. The estimation of FIGARCH parameters using QMLE can be carried out with the help of MFE toolbox of MATLAB (Sheppard, 2009) or G@RCH package of OXMetrics software (Laurent and Peters, 2002).

When estimating the parameters of a FIGARCH model, generally, the value of parameter d is estimated first and one uses these estimates to obtain the estimation of other parameters (Lopes and Mendes, 2006; Härdle and Mungo, 2008). There are several ways to estimate d. Log periodogram regression estimator (GPH) of Geweke and Porter-Hudak (1983) and the Gaussian semiparametric estimator (GPS) of Robinson (1995) are some of them.

Now we will consider the problem of forecasting using a FIGARCH model. Using (10), we have

$$\begin{aligned} h_{t+1} &= \alpha_0(1-\beta(1))^{-1} + \lambda(L)\epsilon_{t+1}^2 \\ &= \alpha_0(1-\beta(1))^{-1} + \lambda_1\epsilon_t^2 + \lambda_2\epsilon_{t-1}^2 + \dots . \end{aligned}$$

The one-step ahead forecast of h_t is given by

$$h_t(1) = \alpha_0(1-\beta(1))^{-1} + \lambda_1\epsilon_t^2 + \lambda_2\epsilon_{t-1}^2 + \dots .$$

Similarly, the two-step ahead forecast is given by

$$h_t(2) = \alpha_0(1-\beta(1))^{-1} + \lambda_1 h_t(1) + \lambda_2\epsilon_t^2 + \dots .$$

In general, the l-step ahead forecast is

$$h_t(l) = \alpha_0(1-\beta(1))^{-1} + \lambda_1 h_t(l-1) + \dots + \lambda_{l-1}h_t(1) + \lambda_l\epsilon_t^2 + \lambda_{l+1}\epsilon_{t-1}^2 + \dots .$$

For all practical purpose, we stop at a large M and this leads to the forecasting equation

$$h_t(l) \approx \alpha_0(1-\beta(1))^{-1} + \sum_{i=1}^{l-1}\lambda_i h_t(l-i) + \sum_{j=0}^{M}\lambda_{l+j}\epsilon_{t-j}^2 .$$

The parameters will have to be replaced by their corresponding estimates.

4 Applications

4.1 Review of Some Applications

There is a large collection of research papers where FIGARCH models are found to be performing better than many of the other conditional heteroscedastic models. Baillie et al. (1996) had applied the FIGARCH to model the volatility in exchange rates, so as Bollerslev and Mikkelsen (1996) and Beine, Laurent, and Lecourt (2002) to stock returns. Banerjee and Sarkar (2006) modeled the volatility in the returns from the National Stock Exchange (NSE) of India, using high frequency intra-day data. Baillie, Han, Myers, and Song (2007) examined the long memory properties of both daily and high frequency intraday futures returns for six commodities. Commodity future returns are found to be well described by a martingale difference sequence with a very low-order moving average (MA) process to represent nonsynchronous trading effects, and also a FIGARCH model for the conditional variance. Antonakakis (2007) investigated the forecasting performance of daily exchange rate volatility in industrialised and developing countries. His study indicated that among all heteroscedastic models, FIGARCH fitted the data better. Also the performance of FIGARCH model in out-of-sample forecasting was superior. Cheong, Isa, and Nor (2008) have studied the volatility in Malaysian stock market returns (KLCI) using a FIGARCH model which allows sudden changes in volatility. Their results revealed that there was structural changes in volatility, especially when a currency crisis happend. Goudarzi (2010), Mukherjee, Sen, and Sarkar (2011) and Sawant and Yadav (2011) have examined the presence of long memory in the stock market return series from Bombay Stock Exchange (BSE). All of them found that FIGARCH model was more appropriate for several stock return series. Crato and Ray (2000) and Jin and Frechette (2004) have studied the long memory effect in the daily volatilities of several agricultural commodity futures returns. Another notable application of FIGARCH models can be found in Q. Li, Tricaud, Sun, and Chen (2007). These authors have attempted to forecast the great salt lake level using a FIGARCH model.

4.2 HDFC Bank Exchange Rate Data

We discuss the modelling of a time series data obtained from HDFC Bank of India on US Dollar- Indian Rupees spot exchange rate for the period January 3, 2000 through January 11, 2011, a total 2797 observations. Figure 1 represents the percentage returns of the exchange rate series. It clearly indicates the volatility in the series during the time period under consideration. The autocorrelation function of the absolute returns has got hyperbolic decay (see Figure 2), indicating the possibility of modelling the series with a long memory model.

We fit the following FIGARCH model to this data set:

$$y_t = 100\log(s_t/s_{t-1}) = \mu + \epsilon_t\,, \qquad \epsilon_t h_t^{-1/2} \overset{iid}{\sim} \mathrm{N}(0,1)$$
$$h_t = \alpha_0 + \beta_1 h_{t-1} + [1 - \beta_1 L - (1-\phi_1 L)(1-L)^d]\epsilon_t^2\,, \qquad t = 1,2,\ldots,2797\,,$$

where s_t is the raw spot exchange rate series and following the standard practice, we concentrate on modeling the daily nominal percentage returns, y_t.

The QMLE's were calculated under the assumption of conditional normality. Robust standard errors are reported in parentheses. The skewness and kurtosis for the standard residuals $\hat{\epsilon}_t\hat{h}_t^{-1/2}$ are denoted by S and K, respectively. The quantities $Q(20)$ and $Q^2(20)$ refer to Ljung-Box portmanteau tests up to 20th-order serial correlation in the standardized and the squared standardized residuals, respectively. The first column of

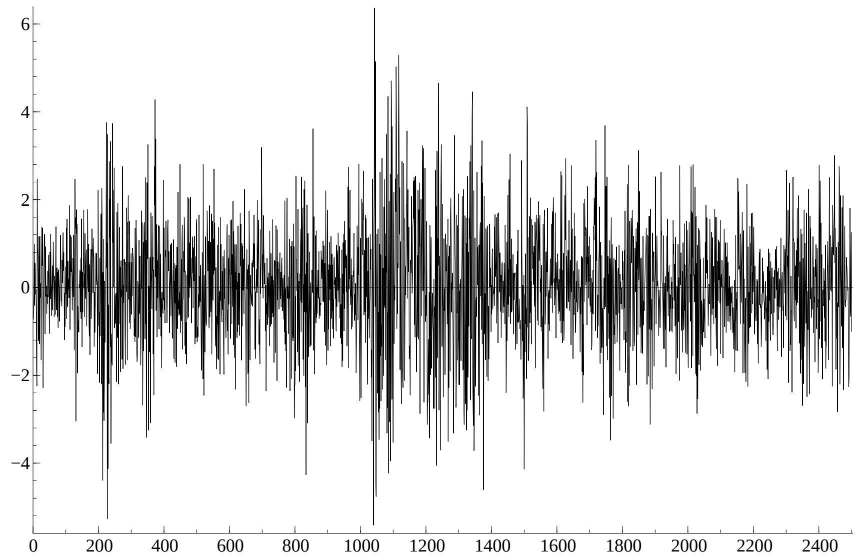

Figure 1: Dollar-Rupees returns

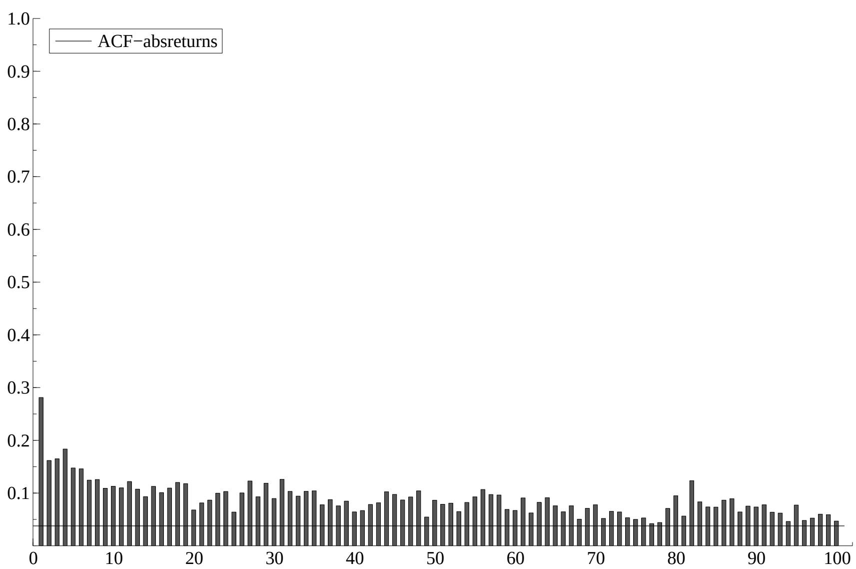

Figure 2: Autocorrelation function (ACF) of absolute returns

$(p, d, q) = (0, 0, 0)$ corresponds to GARCH$(0, 0)$, the second and third to GARCH$(1, 1)$ and IGARCH$(1, 1)$, respectively. Other columns corresponds to the different FIGARCH models. It may be noted that the value of the loglikelihood of the FIGARCH$(1, d, 1)$ model (see Table 1) is marginally higher than for others, indicating the suitability of the FIGARCH$(1, d, 1)$ model.

Using the information criteria computations given in Table 2, we conclude that the FIGARCH $(1, 0.56, 1)$ process fits well for the HDFC bank exchange rate data. Thus, we propose FIGARCH $(1, 0.56, 1)$ as a suitable model for this data set. Table 3 gives the mean

Table 1: Fitted FIGARCH models

(p,d,q)	$(0,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,d,0)$	$(1,d,1)$
μ	0.0006	0.001	0.001	0.001	0.001
	(0.0004)	(0.0003)	(0.0003)	(0.0003)	(0.0003)
α_0	0.0001	0.0001	0.0001	0.0001	0.0001
β_1	–	0.77	0.22	0.26	0.51
		(0.06)	(0.06)	(0.29)	(0.13)
ϕ_1	–	0.197	–	–	0.20
		(0.07)			(0.11)
d	–	–	1.00	0.5	0.56
			(–)	(0.25)	(0.16)
S	-0.23	-0.23	-0.23	-0.23	-0.23
K	12.1	12.03	12.03	12.03	12.03
$Q(20)$	34.14	14.80	14.36	14.98	14.7
$Q(20)^2$	450.534	7.151	8.387	5.69	6.2
Loglikelihood	6443.02	6757.27	6753.78	6761.979	6766.55

Table 2: Information criteria for model selection

(p,d,q)	Akaike	Schwarz	Shibata	Hannan-Quinn
$(0,0,0)$	-4.60566	-4.601410	-4.80566	-4.60413
$(1,0,1)$	-4.82894	-4.820451	-4.82895	-4.82588
$(1,1,0)$	-4.82716	-4.820790	-4.82716	-4.82486
$(1,0.5,0)$	-4.83231	-4.823820	-4.83231	-4.82924
$(1,0.56,1)$	-4.83486	-4.824245	-4.83486	-4.83103

Table 3: Forecasting measure error for conditional variances

(p,d,q)	MSE	RMSE
$(0,0,0)$	3.052e-007	0.00055
$(1,0,1)$	4.962e-007	0.00071
$(1,1,0)$	7.346e-007	0.00085
$(1,0.5,0)$	4.758e-007	0.00069
$(1,0.56,1)$	4.581e-007	0.00068

square error (MSE) and root mean square error (RMSE) corresponding to conditional variance forecasts up to 10 future lags. It may be noted that FIGARCH$(1,0.56,1)$ gives better forecasts.

5 Some Models Related to FIGARCH

5.1 A-FIGARCH Model

Baillie and Morana (2009) have introduced a new long memory volatility process known as Adaptive FIGARCH, or A-FIGARCH. This model is designed to account for both long memory and structural changes in the volatility processes of economic and financial time series. Hence the A-FIGARCH has a stochastic long memory component and a deterministic break process component. The A-FIGARCH(p,d,q,k) process can be derived from the FIGARCH(p,d,q) process by allowing the intercept w in the conditional variance equation to be time varying. The conditional variance equation is given by

$$[1-\beta(L)](h_t-w_t)=[1-\beta(L)-\phi(L)(1-L)^d]\epsilon_t^2\,,$$

where

$$w_t=w_0+\sum_{j=1}^{k}[\gamma_j\sin(2\pi jt/T)+\delta_j\cos(2\pi jt/T)]\,.$$

This model has components with long memory effect and a time-varying intercept. It allows for breaks, cycles and changes in drift. Eventhough w_t is smooth, it is capable of approximating abrupt regime switching. Inference related to the parameters of the A-FIGARCH can be done by the method of QMLE.

Nasr, Boutahr, and Trabelsi (2010) have introduced a new model namely, fractionally integrated time-varying GARCH (FITVGARCH) to capture both long memory and structural changes in the volatility process. The A-FIGARCH model allows the intercept to be a slowly varying function, but this new model allows all the parameters in the conditional variance equation of the FIGARCH to be time dependent. More presicely, the change in the parameters over time is assumed to be smooth using a logistic smooth transition function.

5.2 HYGARCH Model

Davidson (2004) introduced HYGARCH (hyperbolic GARCH) model which is again closely related to the class of FIGARCH. It is known that a GARCH(p,q) model can be rewritten as an ARMA model in squares:

$$\phi(L)\epsilon_t^2=\alpha_0+\beta(L)\nu_t\,,$$

where $\nu_t=\epsilon_t^2-h_t$. By rearranging the terms we can write

$$h_t=\frac{\alpha_0}{\beta(1)}+\left(1-\frac{\phi(L)}{\beta(L)}\right)\epsilon_t^2=\alpha_0+\lambda(L)\epsilon_t^2\,.$$

Then, in an IGARCH(p,q) model we have

$$\lambda(L)=\left(1-\frac{\phi(L)}{\beta(L)}\right)(1-L)$$

and for a FIGARCH(p, d, q) model, it is

$$\lambda(L) = \left(1 - \frac{\phi(L)}{\beta(L)}\right)(1 - L)^d\,, \qquad 0 < d < 1\,.$$

The FIGARCH model is a generalization of the IGARCH model for hyperbolic lag weights. The characterization of the FIGARCH model as an intermediate case between the stable GARCH and IGARCH is misleading. In fact, it has more memory than either of these models but behaves oddly because of the restriction of unit amplitude. Therefore, in this context, Davidson (2004) introduced a new model, for which he considered amplitude and long memory as separate phenomena. The unit-amplitude restriction can be tested in this model. The model could address the long memory character without behaving oddly when d approximated to one. This model is called "Hyperbolic GARCH" or HYGARCH and it is given by

$$\lambda(L) = \left(1 - \frac{\phi(L)}{\beta(L)}\right)(1 + \alpha((1 - L)^d - 1))\,,$$

where $\alpha \geq 0$ is the amplitude parameter and $d \geq 0$. Davidson (2004) suggested the use of QMLE method to estimate the HYGARCH parameters.

5.3 ST-FIGARCH Model

Kilic (2011) introduced the smooth transition autoregressive FIGARCH (ST-FIGARCH) model, which can jointly consider the long memory and nonlinearity in the conditional volatility process. The ST-FIGARCH model generalizes the FIGARCH model by allowing nonlinear dynamics and asymmetry by way of introducing a smooth transition specification for the conditional variance. The ST-FIGARCH model is capable of accommodating smooth changes both in the amplitude of volatility clusters as well as asymmetry in conditional volatility in a relatively parsimonious way. Such dynamics cannot be modeled by standard FIGARCH. ST-FIGARCH allows the conditional variance to depend on the evolution of the variable, called transition. Depending on the sign and amplitude of the transition variable, conditional variance can evolve smoothly between low and high volatility regimes. As in other models, here also the first order of the model was introduced. The smooth transition FIGARCH$(1, d, 1)$ model is defined as

$$(1 - \phi L)(1 - L)^d \epsilon_t^2 = \alpha_0 + [1 - \beta(1 - G(z_{t-s}, \gamma, c))L - \beta^* G(z_{t-s}, \gamma, c)L]\nu_t\,,$$

where $\nu_t = \epsilon_t^2 - h_t$, $0 < d < 1$, β and β^* are the volatility dynamics parameters, and

$$G(z_{t-s}, \gamma, c) = \frac{1}{1 + \exp(-\gamma(z_{t-s} - c))}$$

is the logistic transition function with transition variable s-period lagged z, where s is called the delay parameter and c is the location or threshold parameter. The transition parameter γ is assumed to be positive for identification purposes and characterizes the speed of transition between extreme regimes.

A useful characteristics of ST-FIGARCH is that the researcher can choose z according to his or her problem. Possible choices may include time (which may be useful if one thinks that conditional volatility may have smooth changing shifts), functions of past values of the return series and past values of unobserved shocks. One such example would be news that may cause smooth changes in the volatility dynamics in exchange rates and stock markets. Estimation and inference for the parameters of this model can be carried out using the method of QMLE suggested by Kilic (2011). A formal proof of consistency and asymptotic normality of the QMLE are yet to be investigated fully in this case.

5.4 Asymmetric FIGARCH Model

Many a time, empirical findings indicate that large negative returns are followed by larger increases in volatility than equally for large positive returns. This asymmetric effect leads to the introduction of several types of asymmetric volatility models. The long memory in variance and asymmetry facts in financial markets has been extensively discussed in literature. There have been two separate efforts to resolve both these features. The new class of asymmetric fractionally integrated family of generalized autoregressive conditional heteroscedastic (ASYMM–FIFGARCH) models introduced by Hwang (2001), combine the long memory and asymmetry;

$$\epsilon_t = h_t^{1/2} z_t \,,$$

$$h_t^{\lambda/2} = \frac{\alpha_0}{1-\beta_1} + \left[1 - \frac{(1-\phi_1 L)(1-L)^d}{1-\beta_1 L}\right] f^{\nu}(\epsilon_t) h_t^{\lambda/2} \,, \tag{13}$$

$$f(\epsilon_t) = \left|\frac{\epsilon_t}{\sigma_t} - b\right| - c\left(\frac{\epsilon_t}{\sigma_t} - b\right) \,, \qquad |c| \le 1 \,, \tag{14}$$

with $z_t \overset{iid}{\sim} \mathrm{D}(0,1)$, where $\mathrm{D}(0,1)$ represents some specific distribution with mean zero and variance one. Here, b and c respectively denote the shift and rotation of the news impact curve. For example, it can be normal, Student-t, or more flexible distributions such as NIG, variance-gamma (VG), generalized hyperbolic etc. Ruiz and Perez (2003) claimed that this model is badly misspecified when $\lambda = 0$, a case for which the conditional standard deviation is not defined. Therefore Ruiz and Perez (2003) modified the model given in (13) as

$$\epsilon_t = h_t^{1/2} z_t$$

$$(1-\phi_1 L)(1-L)^d \frac{h_t^{\lambda/2} - 1}{\lambda} = \alpha_0^* + \alpha(1+\psi L) h_{t-1}^{\lambda/2} [f^{\nu}(z_{t-1}) - 1] \,,$$

where $f(\cdot)$ is the same as in (14). When $\nu = \lambda = 2$ and $b = c = 0$, we reach exactly the FIGARCH model.

6 Concluding Remarks

We have made an attempt to present a brief review on FIGARCH models, its properties, applications and few models which are related FIGARCH models. The probabilistic and inferential aspects of such models are not completely investigated. Most of the developments in the literature are related to the empirical investigations. The stationarity and ergodicity of a general FIGARCH(p, d, q) model is yet to be established fully.

From the inference point of view also, this model is not completely exploited by the researchers. The asymptotic theory of the quasi-maximum likelihood is not completely resolved. Several non/semi parametric methods are yet to be tried on this model. One another problem of interest would be treating w_t of an A-FIGARCH model as a stochastic function. e.g., $w_t = w + u_t$, where the u_t's are iid random variables. Inference and significance tests ($\sigma_u^2 = 0$) are some of the future related directions of research.

References

Abramowitz, M., and Stegun, I. A. (1965). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. New York: Dover Publications, Inc.

Andersen, T. G., Davis, R. A., Kreis, J. P., and Mikosch, T. (2009). *Handbook of Financial Time Series*. Berlin: Springer Verlag.

Anderson, J. (2001). On the normal inverse Gaussian stochastic volatility model. *Journal of Business and Economic Statistics*, *19*, 44-54.

Antonakakis, N. (2007). *Forecasting nominal exchange rate volatility in developing countries.* Department of Economics, University of Strathclyde.

Baillie, R. T., Bollerslev, T., and Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *74*, 3-30.

Baillie, R. T., Han, Y. W., Myers, R. J., and Song, J. (2007). Long memory models for daily and high frequency commodity futures returns. *The Journal of Futures Markets*, *27*, 643-668.

Baillie, R. T., and Morana, C. (2009). Modeling long memory and structural breaks in conditional variances: An adaptive FIGARCH approach. *Journal of Econometric Dynamics and Control*, *33*, 1577-1592.

Banerjee, A., and Sarkar, S. (2006). *Modeling daily volatility of the Indian stock market using intra-day data.* Working paper WPSNO.588. Indian Institute of Management, Calcutta.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modeling. *Scandinavian Journal of Statistics*, *24*, 1-13.

Beine, M., Laurent, S., and Lecourt, C. (2002). Accounting for conditional leptokurtosis and closing days effects in FIGARCH models of daily exchange rates. *Applied Financial Economics*, *12*, 589-600.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics*, *31*, 307-328.

Bollerslev, T., and Mikkelsen, H. O. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, *73*, 151-184.

Bollerslev, T., and Wooldridge, J. M. (1992). Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances. *Econometric Reviews*, *11*, 143-172.

Bordignon, S., Caporin, M., and Lisi, F. (2004). *A seasonal fractionally integrated GARCH model.* Working paper. University of Padova, Italy.

Bougerol, P., and Picard, N. (1992). Stationarity of GARCH processes and of some nonnegative time series. *Journal of Econometrics*, *52*, 15-128.

Caporin, M. (2003). Identification of long memory in GARCH models. *Statistical Methods and Applications*, *12*, 133-1510.

Cheong, C. W., Isa, Z., and Nor, A. H. S. M. (2008). Fractionally integrated time-varying volatility under structural break: Evidence from Kuala Lumpur composite index. *Sains Malaysiana*, *37*, 405-411.

Conrad, C., and Haag, B. (2006). Inequality constraints in the fractionally integrated GARCH model. *Journal of Financial Econometrics*, *4*, 413-449.

Crato, N., and Ray, B. K. (2000). Memory in returns and volatilities of futures contracts. *Journal of Futures Markets*, *20*, 525-5430.

Davidson, J. (2004). Moment and memory properties of linear conditional heteroscedasticity models, and a new model. *Journal of Business and Economic Statistics*, *22*, 16-190.

Douc, R., Roueff, F., and Soulier, P. (2008). On the existence of some ARCH(∞) processes. *Stochastic Processes and Applications*, *118*, 755-761.

Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica*, *50*, 987-10080.

Engle, R. F., and Bollerslev, T. (1986). Modelling the persistence of conditional variances. *Econometric Reviews*, *5*, 1-500.

Francq, C., and Zakoian, J. M. (2010). *GARCH Models: Structure, Statistical Inference and Financial Applications*. New York: John Wiley.

Geweke, J., and Porter-Hudak, S. (1983). The estimation and application of long memory time series models. *Journal of Time Series Analysis*, *4*, 221-238.

Giraitis, L., Kokoszka, P., and Leipus, R. (2000). Stationarity ARCH models: Dependence structure and central limit theorem. *Econometric Theory*, *16*, 3-22.

Giraitis, L., Leipus, R., and Surgailis, D. (2009). ARCH(∞) models and long memory. In T. G. Anderson, R. A. Davis, J. Kreis, and T. Mikosch (Eds.), *Handbook of Financial Time Series* (p. 71-84). Berlin: Springer Verlag.

Goudarzi, H. (2010). Modeling long memory in the Indian stock market using fractionally integrated EGARCH model. *International Journal of Trade, Economics and Finance*, *1*, 231-237.

Granger, C. W. J. (1980). Long memory relationships and the aggregation of dynamic models. *Journal of Econometrics*, *14*, 227-238.

Granger, C. W. J. (1981). Some properties of time series data and their use in econometric model specification. *Journal of Econometrics*, *16*, 121-130.

Granger, C. W. J., and Joyeux, R. (1980). An introduction to long memory time series models and fractional differencing. *Journal of Time Series Analysis*, *1*, 15-39.

Härdle, W. K., and Mungo, J. (2008). Long memory persistence in the factor of implied volatility dynamics. *International Research Journal of Finance and Economics*, *18*, 213-230.

Hosking, J. R. M. (1981). Fractional differencing. *Biometrika*, *68*, 165-176.

Hwang, Y. (2001). Asymmetric long memory GARCH in exchange return. *Econometric*

Letters, *73*, 1-5.

Jensen, M. B., and Lunde, A. (2001). The NIG-S and ARCH model: a fat tailed, stochastic, and autoregressive conditional heteroscedastic volatility model. *Econometrics Journal*, *4*, 319-342.

Jin, H. J., and Frechette, D. (2004). Fractional integration in agricultural futures price volatilities. *American Journal of Agricutural Economics*, *86*, 432-443.

Kazakevicius, V., and Leipus, R. (2002). On stationarity in the ARCH(∞) model. *Econometric Theory*, *18*, 1-16.

Kilic, R. (2007). Conditionl volatility and distribution of exchange rates: GARCH and FIGARCH models with NIG distribution. *Studies in Nonlinear Dynamics and Econometrics*, *11*, 1-31.

Kilic, R. (2011). Long memory and nonlinearity in conditional variances: A smooth transition FIGARCH model. *Journal of Empirical Finance*, *18*, 7368.

Laurent, S., and Peters, J. P. (2002). G@RCH 2.2: An OX package for estimating and forecasting various ARCH models. *Journal of Economic Surveys*, *16*, 447-4850.

Li, C. S., and Xiao, Q. X. (2011). Structural break in persistence of European stock market: Evidence from panel GARCH model. *International Journal of Intelligent Information Processing*, *2*, 40-48.

Li, Q., Tricaud, C., Sun, R., and Chen, Y. (2007). Great salt lake surface level forecasting using FIGARCH modeling. In *Proceeding of the ASME 2007 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference.* USA: IFETC/CIE.

Lopes, S. R. C., and Mendes, B. V. M. (2006). Bandwidth selection in classical and robust estimation of long memory. *International Journal of Statistics and Systems*, *1*, 107-190.

Mikosch, M., and Starica, C. (2003). Long range dependence effects and ARCH modelling. In M. S. Taqqu, G. Oppenhein, and P. Doukhan (Eds.), *Long-Range Dependence: Theory and Applications* (p. 539-549). Boston: Birkhauser.

Mukherjee, I., Sen, C., and Sarkar, A. (2011). Long memory in stock returns: Insights from the Indian market. *The International Journal of Applied Economics and Finance*, *5*, 62-74.

Nasr, A. B., Boutahr, M., and Trabelsi, A. (2010). Fractionally integrated time varying GARCH model. *Statistical Methods and Applications*, *19*, 399-430.

Nelson, D. B. (1990). Stationarity and persistence in the GARCH(1, 1) model. *Econometric Theory*, *6*, 318-334.

Rapach, D. E., and Strauss, J. K. (2008). Structural breaks and GARCH models of exchange rate volatility. *Journal of Applied Econometrics*, *23*, 65-90.

Robinson, P. M. (1995). Gaussian semiparametric estimation of long-range dependence. *Annals of Statistics*, *23*, 1630-1661.

Rohan, N. (2009). *Conditional Heteroscedastic Time Series Models with Asymmetry and Structural Breaks* [Unpublished M. Phil. Thesis]. Department of Statistics, University of Pune, India.

Rohan, N., and Ramanathan, T. V. (2012). *Asymmetric volatility models with structural breaks.* (To appear in Communications in Statistics-Simulations and Computations)

Ruiz, E., and Perez, A. (2003). Asymmetric long memory GARCH: A reply to Hwang's model. *Economic Letters*, *73*, 415-4220.

Sawant, A., and Yadav, R. (2011). *Long Memory Time Series Modelling for BSE Stock Returns.* Department of Statistics, University of Pune, India. (Project report)

Sheppard, K. (2009). MFE MATLAB Function Reference. Financial Econometrics [Computer software manual].

Smith, D. R. (2008). Testing for structural breaks in GARCH models. *Applied Financial Economics*, *18*, 845-862.

Taylor, S. J. (1986). *Modelling Financial Time Series*. New York: Wiley.

Weiss, A. (1986). Asymptotic theory for ARCH models: Estimation and testing. *Econometric Theory*, *2*, 101-103.

Xekalaki, E., and Degiannakis, S. (2010). *ARCH Models for Financial Applications*. New York: Wiley.

Zaffaroni, P. (2004). Stationarity and memory of ARCH(∞) models. *Econometric Theory*, *20*, 147-160.

Authors' address:
Maryam Tayefi and T. V. Ramanathan
Department of Statistics and Centre for Advanced Studies
University of Pune
411 007 India
E-mail: `maryam_tayefi@yahoo.com`