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On the Szekely-Mori Asymmetry Criterion Statistics for Binary Vectors with Independent Components

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Abstract: For random binary vectors the first two moments and limit distributions of statistics in a recently proposed by Székely and Móri criterion of asymmetry of a distribution are investigated.

Keywords: Test, Symmetry of Distribution, Random Binary Vectors, Limit Distribution.

1 Introduction

Let $X, X_1, X_2, \dots$ be i.i.d. R^d-valued random vectors and $\|X\|$ denotes a Euclidian norm of vector X. It was shown by Szekely and Mori (2001) that $\mathrm{E}(\|X_1+X_2\|-\|X_1-X_2\|) \geq 0$ and that $\mathrm{E}(\|X_1+X_2\|-\|X_1-X_2\|) = 0$ if and only if X is symmetrically distributed (i.e., if the distributions of X and $-X$ coincide).

A sequence of statistics

$$T_n = T_n(X_1, \dots, X_n) = \frac{\sum_{1\leq i<j\leq n} (\|X_i + X_j\| - \|X_i - X_j\|)}{\sum_{1\leq i\leq n} \|X_i\|}$$

was proposed by Szekely and Mori (2001) as a base of a consistent test for symmetry against general alternatives. According to Szekely and Mori (2001) if $\mathrm{E}(\|X\|) < \infty$ and $x(\alpha) = (\Phi^{-1}(1-\alpha/2))^2$ then

$$\sup_{H_0} \lim_{n\to\infty} \Pr\{1 + T_n \geq x(\alpha)\} = \alpha, \tag{1}$$

where H_0 is the set of all symmetrical distributions in R^d.

Here the equality holds for two-point symmetric distributions where $\Pr\{X_1 = a\} = \Pr\{X_1 = -a\} = 1/2$ for some $a \in R^d \backslash \{0\}$. Hence,

$$\Pr\left\{T_n = \frac{1}{n}(n-2m)^2 - 1\right\} = \frac{1}{2^n} C_n^m, \qquad m = 0, 1, \dots, n,$$

and $\mathrm{E}(T_n) = 0$, $\mathrm{D}(T_n) = 2(n-1)/n$. According to the deMoivre-Laplace theorem

$$\Pr\{T_n + 1 \leq x\} \to \Phi(\sqrt{x}) - \Phi(-\sqrt{x}), \qquad n \to \infty,$$

which corresponds to (1).

2 Main Results

We will consider the case when the distribution of the vector X is concentrated on a vertex set $V_d = \{B = (b_1, \ldots, b_d) : b_j \in \{-1, +1\}, j = 1, \ldots, d\}$ of d-dimensional cube (so $\Pr\{\|X\| = \sqrt{d}\} = 1$ and T_n is a U-statistics in this case).

If $X = (x_1, \ldots, x_d)$ is uniformly distributed on B_d then $x_1, \ldots, x_d$ are independent and $\Pr\{x_i = -1\} = \Pr\{x_i = 1\} = 1/2$. If the random vector Y is independent and identically distributed with X then

$$\mathrm{E}(\|X + Y\| - \|X - Y\|) = 0. \tag{2}$$

Theorem 1. *If random vectors $X = (x_1, \ldots, x_d)$, $Y = (y_1, \ldots, y_d)$ are independent and uniformly distributed on V_d, $d \geq 1$, then*

$$\mathrm{D}(\|X + Y\| - \|X - Y\|) = \frac{1}{2^{d-3}} \sum_{m=0}^{d} C_d^m \left(\frac{d}{2} - \sqrt{m(d-m)} \right) = 2 + \frac{\theta_d}{d},\ \theta_d \in \left[\tfrac{1}{2}, 6\right].$$

PROOF. In view of (2)

$$\mathrm{D}\left(\|X + Y\| - \|X - Y\|\right) = \mathrm{E}\left(\sqrt{\sum\nolimits_{j=1}^{d} (x_j + y_j)^2} - \sqrt{\sum\nolimits_{j=1}^{d} (x_j - y_j)^2} \right)^2$$
$$= 2\mathrm{E}\left(\sum\nolimits_{j=1}^{d} (x_j^2 + y_j^2) \right) - 2\mathrm{E}\left(\sqrt{\sum\nolimits_{j=1}^{d} (x_j + y_j)^2} \sqrt{\sum\nolimits_{j=1}^{d} (x_j - y_j)^2} \right).$$

In our case $\mathrm{E}\left(\sum_{j=1}^{d} (x_j^2 + y_j^2) \right) = 2d$. Let us consider sets $A_m = \{(B, C) \in V_d \times V_d : |\{k : b_k = c_k\}| = m\}$ for $m = 0, \ldots, d$. If $(B, C) \in A_m$ then

$$\sqrt{\sum\nolimits_{j=1}^{d} (b_j + c_j)^2} \sqrt{\sum\nolimits_{j=1}^{d} (b_j - c_j)^2} = 4\sqrt{m(d-m)}.$$

The set A_m consists of $C_d^m 2^d$ elements, the set of all possible pairs (B, C) consists of 2^{2d} elements. Consequently,

$$\mathrm{E}\left(\sqrt{\sum\nolimits_{j=1}^{d} (x_j + y_j)^2} \sqrt{\sum\nolimits_{j=1}^{d} (x_j - y_j)^2} \right) = \frac{1}{2^{d-2}} \sum_{m=0}^{d} C_d^m \sqrt{m(d-m)}$$

and

$$\mathrm{D}(\|X_1 + X_2\| - \|X_1 - X_2\|) = 4d - \frac{1}{2^{d-3}} \sum_{m=0}^{d} C_d^m \sqrt{m(d-m)}$$
$$= \frac{1}{2^{d-3}} \sum_{m=0}^{d} C_d^m \left(\frac{d}{2} - \sqrt{m(d-m)} \right) = 8\mathrm{E}\left(\frac{d}{2} - \sqrt{\xi(d-\xi)} \right),$$

where ξ is a random variable with the binomial distribution $\mathrm{Bin}(d, 1/2)$.

It is easy to check that $\mathrm{E}(\xi - d/2)^2 = d/4$, $\mathrm{E}(\xi - d/2)^4 = d(3d-2)/16$ and

$$1 - \frac{x}{2} - \frac{x^2}{2} \le \sqrt{1-x} \le 1 - \frac{x}{2} - \frac{x^2}{8} \qquad \text{for } 0 \le x \le 1.$$

So

$$\frac{d}{2} - \sqrt{x(d-x)} = \frac{d}{2}\left(1 - \sqrt{1 - \left(\frac{x-d/2}{d/2}\right)^2}\right)$$

$$= \frac{d}{2}\left(\frac{1}{2}\frac{(x-d/2)^2}{(d/2)^2} + \theta\frac{(x-d/2)^4}{(d/2)^4}\right) = \frac{(x-d/2)^2}{d} + 8\theta\frac{(x-d/2)^4}{d^3}, \quad \theta \in \left[\frac{1}{8}, \frac{1}{2}\right],$$

and

$$\mathrm{D}(\|X_1 + X_2\| - \|X_1 - X_2\|) = 8\mathrm{E}\left(\tfrac{d}{2} - \sqrt{\xi(d-\xi)}\right)$$

$$= 8\mathrm{E}\left(\frac{(\xi-d/2)^2}{d} + 8\theta\frac{(\xi-d/2)^4}{d^3}\right) = 2 + \frac{\theta_d}{d}, \qquad \theta_d \in \left[\frac{1}{2}, 6\right].$$

(The set of possible values of θ_d was widened to be valid for all $d \ge 1$.) Theorem 1 is proved.

Theorem 2. *If random variables $X = (x_1, \ldots, x_d)$, $Y = (y_1, \ldots, y_d)$ are independent and uniformly distributed on V_d then for all $t \in (-\infty, \infty)$*

$$\Pr\{(\|X+Y\| - \|X-Y\|) \le t\} \to \Phi\left(\frac{t}{\sqrt{2}}\right), \qquad d \to \infty.$$

PROOF. The distribution of $\|X+Y\| - \|X-Y\|$ coincides with that of $\eta = 2(\sqrt{\xi_d} - \sqrt{d-\xi_d})$, where ξ_d has a binomial distribution $\mathrm{Bin}(d, 1/2)$.

Notice that the function $u(x) = \sqrt{x} - \sqrt{d-x}$ is increasing on $[0, d]$. Therefore,

$$F_\eta(x) = \Pr\{\eta \le x\} = \sum_{m:u(m)\le x} p_m, \qquad \text{where} \quad p_m = \Pr\{\xi_d = m\} = \frac{1}{2^d}C_d^m.$$

It is easy to check that $k(t) \stackrel{\text{def}}{=} \max\{x : u(x) \le t\} = \frac{d}{2} + \frac{t}{2}\sqrt{\frac{d}{2}}\sqrt{1 - \frac{t^2}{8d}}$. Consequently,

$$F_\eta(t) = \Pr\{\eta \le t\} = \sum_{m=0}^{k(t)} p_m = \Pr\{\xi_d \le k(t)\}$$

$$= \Pr\left\{\frac{\xi_d - d/2}{\sqrt{d}/2} \le \frac{t}{\sqrt{2}}\sqrt{1 - \frac{t^2}{8d}}\right\} \to \Phi\left(\frac{t}{\sqrt{2}}\right), \qquad d \to \infty,$$

for each $t \in (-\infty, \infty)$ due to the deMoivre-Laplace theorem.

By means of Theorem 1 we may find two first moments of the U-statistics T_n for uniform distribution on V_d. We have

$$\mathrm{E}(T_n) = \frac{(n-1)}{2\sqrt{d}}\mathrm{E}(\|X_1 + X_2\| - \|X_1 - X_2\|) = 0.$$

Since for independent vectors $X_1, X_2, \dots$ with symmetrical distribution on V_d for any $a \in V_d$ we have

$$\begin{aligned} &\mathrm{E}(\|a+X_i\| - \|a-X_i\|) = 0, \qquad\qquad (3)\\ &\mathrm{E}(\|a+X_i\| - \|a-X_i\|)(\|a+X_j\| - \|a-X_j\|) = 0, \qquad i \neq j, \end{aligned}$$

it follows that

$$\mathrm{cov}(\|X_i+X_j\| - \|X_i-X_j\|, \|X_k+X_l\| - \|X_k-X_l\|) = 0$$

for all $1 \le i < j$, $1 \le k < l$, $(i,j) \neq (k,l)$. So

$$\begin{aligned} \mathrm{D}(T_n) &= \frac{1}{n^2 d}\mathrm{D}\left(\sum_{1\le i<j\le n} (\|X_i+X_j\| - \|X_i-X_j\|)\right)\\ &= \frac{n-1}{2nd}\mathrm{D}(\|X_1+X_2\| - \|X_1-X_2\|) \to \frac{1}{d}, \qquad d \to \infty. \end{aligned}$$

Due to (3) U-statistics T_n are degenerate ones. Applying the results of Gregory (1977) (see also Korol'uk and Borovskih (1989)) to our case we obtain that if $d = \mathrm{const}$ and $n \to \infty$ then distributions of U-statistics T_n converge to the distribution of $\sum_{k=1}^{2^d} c_k \nu_k^2 - 1$, where $\nu_1, \nu_2, \dots$ are independent random variables with standard Gaussian distribution, $c_k \ge 0$, $\sum c_k = 1$ and the coefficients c_k are the eigenvalues of operator $S : f(x) \to \mathrm{E}(\|X_1 + x\| - \|X_1 - x\|) f(X_1)$ in $L^2(V_d)$ (see Szekely and Mori, 2001). The exact formulas for these coefficients in the case of general d are under investigation.

This results may be used to construct a goodness-of-fit test for generators of random or pseudorandom bits.

Now we consider a class of nonuniform distributions on V_d corresponding to random vectors with independent components.

Theorem 3. *If random vectors $X = (x_1, \dots, x_d)$, $Y = (y_1, \dots, y_d)$ with values in V_d are independent identically distributed with independent components,*

$$\Pr\{x_j = 1\} = \tfrac{1}{2} + \varepsilon_j^{(d)}, \;\; \Pr\{x_j = -1\} = \tfrac{1}{2} - \varepsilon_j^{(d)}, \;\; |\varepsilon_j^{(d)}| < \tfrac{1}{2}, \quad j = 1, \dots, d,$$

if $d \to \infty$ and for some $\delta > 0$

$$a_d \overset{\text{def}}{=} \frac{4}{d} \sum_{j=1}^{d} \left(\varepsilon_j^{(d)}\right)^2 < 1 - \delta \qquad \textit{for all } d,$$

then the distribution of $\|X+Y\| - \|X-Y\|$ is asymptotically normal with parameters

$$\left(\frac{2a_d\sqrt{2d}}{\sqrt{1-a_d} + \sqrt{1+a_d}}, (1-b_d)\frac{1+\sqrt{1-a_d^2}}{1-a_d^2}\right), \quad \textit{where } b_d \overset{\text{def}}{=} \frac{16}{d}\sum_{j=1}^{d} \left(\varepsilon_j^{(d)}\right)^4 < a_d.$$

PROOF. Note that $\|X+Y\|^2 = \sum_{j=1}^d (x_j + y_j)^2 = 4\xi_d$, $\|X-Y\|^2 = 4d - \|X+Y\|^2$, where $\xi_d = \xi_d(\varepsilon_1, \dots, \varepsilon_d)$ is the sum of d independent indicators:

$$\xi_d = \sum_{j=1}^{d} \eta_j, \quad \eta_j = I(x_j = y_j), \quad \Pr\{\eta_j = 1\} = \tfrac{1}{2} + 2\left(\varepsilon_j^{(d)}\right)^2, \quad j = 1, \dots, d.$$

So,

$$\mathrm{E}(\xi_d) = \tfrac{d}{2} + 2\sum_{j=1}^{d}\left(\varepsilon_j^{(d)}\right)^2 = \tfrac{d}{2}(1+a_d), \quad \mathrm{D}(\xi_d) = \tfrac{d}{4} - 4\sum_{j=1}^{d}\left(\varepsilon_j^{(d)}\right)^4 = \tfrac{d}{4}(1-b_d),$$

$$\sum_{j=1}^{d}\mathrm{E}(|\eta_j - \mathrm{E}(\eta_j)|^3) < \tfrac{d}{8}\,.$$

Therefore

$$\begin{aligned}\Pr\{\|X+Y\| - \|X-Y\| \le x\} &= \Pr\left\{\sqrt{\xi_d} - \sqrt{d-\xi_d} \le \tfrac{x}{2}\right\} \\ &= \Pr\left\{\sqrt{\tfrac{1}{d}\xi_d} - \sqrt{1-\tfrac{1}{d}\xi_d} \le \tfrac{x}{2\sqrt{d}}\right\}.\end{aligned} \tag{4}$$

It follows from Lyapunov's theorem and conditions of Theorem 3 that $\frac{1}{d}\xi_d$ is asymptotically normal with parameters $(\frac{1}{2}(1+a_d), \frac{1}{4d}(1-b_d))$. Because the derivative of the function $s(x) = \sqrt{x} - \sqrt{1-x}$ is strictly positive and bounded on $\left[\frac{1}{2}, 1-\delta\right]$, the random variable $s(\frac{1}{d}\xi_d) = \sqrt{\frac{1}{d}\xi_d} - \sqrt{1-\frac{1}{d}\xi_d}$ is asymptotically normal with parameters

$$\big(s(\tfrac{1}{d}\mathrm{E}(\xi_d)), (s'(\tfrac{1}{d}\mathrm{E}(\xi_d)))^2\mathrm{D}(\tfrac{1}{d}\xi_d)\big) = \left(\frac{a_d\sqrt{2}}{\sqrt{1-a_d}+\sqrt{1+a_d}}, \frac{1+\sqrt{1-a_d^2}}{1-a_d^2}\frac{1-b_d}{4d}\right). \tag{5}$$

Consequently, the random variable $2(\sqrt{\xi_d} - \sqrt{d-\xi_d})$ is asymptotically normal with parameters

$$\left(\frac{2a_d\sqrt{2d}}{\sqrt{1-a_d}+\sqrt{1+a_d}}, \frac{1+\sqrt{1-a_d^2}}{1-a_d^2}(1-b_d)\right),$$

and Theorem 3 is proved.

Theorem 2 is a particular case of Theorem 3, but its statement is simpler.

Theorem 4. *If the conditions of Theorem 3 are satisfied then there exists a constant* $C = C(a_d) < \infty$ *such that*

$$\left|\mathrm{E}(\|X+Y\| - \|X-Y\|) - \frac{2a_d\sqrt{2d}}{\sqrt{1-a_d}+\sqrt{1+a_d}}\right| < \frac{C}{\sqrt{d}}, \tag{6}$$

and

$$\mathrm{D}(\|X+Y\| - \|X-Y\|) = \frac{1+\sqrt{1-a_d^2}}{1-a_d^2}(1-b_d+o(1)), \qquad d\to\infty. \tag{7}$$

PROOF. We will use notations introduced in the proof of Theorem 3. According to (4)

$$\mathrm{E}(\|X+Y\| - \|X-Y\|) = 2\sqrt{d}\mathrm{E}s(\tfrac{1}{d}\xi_d), \qquad s(x) = \sqrt{x} - \sqrt{1-x}. \tag{8}$$

The function $s(x)$, $x \in [0,1]$, has quadratic lower and upper bounds:

$$\begin{aligned} &s(\tfrac{1}{d}\mathrm{E}(\xi_d)) + s'(\tfrac{1}{d}\mathrm{E}(\xi_d))(x - \tfrac{1}{d}\mathrm{E}(\xi_d)) - C_1(x - \tfrac{1}{d}\mathrm{E}(\xi_d))^2 \le s(x) \le \\ &\le s(\tfrac{1}{d}\mathrm{E}(\xi_d)) + s'(\tfrac{1}{d}\mathrm{E}(\xi_d))(x - \tfrac{1}{d}\mathrm{E}(\xi_d)) + C_2(x - \tfrac{1}{d}\mathrm{E}(\xi_d))^2,\end{aligned} \tag{9}$$

where

$$C_1 = \frac{1 + s(\frac{1}{d}\mathrm{E}(\xi_d)) - s'(\frac{1}{d}\mathrm{E}(\xi_d))\frac{1}{d}\mathrm{E}(\xi_d)}{(\frac{1}{d}\mathrm{E}(\xi_d))^2},$$
$$C_2 = \frac{1 - s(\frac{1}{d}\mathrm{E}(\xi_d)) - s'(\frac{1}{d}\mathrm{E}(\xi_d))(1 - \frac{1}{d}\mathrm{E}(\xi_d))}{(1 - \frac{1}{d}\mathrm{E}(\xi_d))^2}.$$

By means of these estimates we obtain

$$\left|\mathrm{E}(s(\tfrac{1}{d}\xi_d)) - s(\tfrac{1}{d}\mathrm{E}(\xi_d))\right| \le \max\{C_1, C_2\}\mathrm{D}(\tfrac{1}{d}\xi_d) < \max\{C_1, C_2\}\tfrac{1}{4d}. \tag{10}$$

Inequality (6) is a consequence of (8), (10) and $\mathrm{E}(\frac{1}{d}\xi_d) = \frac{1+a_d}{2}$.

It follows from Theorem 3 that there exists a sequence $\{\alpha_d\}$ such that $\alpha_d \to 0$ as $d \to \infty$ and $\mathrm{D}(\|X+Y\| - \|X-Y\|) = 4d\mathrm{D}(s(\frac{1}{d}\xi_d)) \ge (1-\alpha_d)(1-b_d)\frac{1+\sqrt{1-a_d^2}}{1-a_d^2}$. To obtain upper bounds we use (9) as follows:

$$\begin{aligned}\mathrm{D}(s(\tfrac{1}{d}\xi_d)) &\le \mathrm{E}(s(\tfrac{1}{d}\xi_d) - s(\mathrm{E}(\tfrac{1}{d}\xi_d)))^2 \\ &= \mathrm{E}\left(s'(\mathrm{E}(\tfrac{1}{d}\xi_d))(\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d)) + C^*\theta(\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d))^2\right)^2,\end{aligned}$$

where $C^* = \max\{C_1, C_2\}$ and θ is a random variable, $\Pr\{|\theta| \le 1\} = 1$. Therefore,

$$\begin{aligned}\mathrm{D}(s(\tfrac{1}{d}\xi_d)) \le{}& (s'(\mathrm{E}(\tfrac{1}{d}\xi_d)))^2\mathrm{E}(\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d))^2 + 2C^*s'(\mathrm{E}(\tfrac{1}{d}\xi_d))\mathrm{E}(|\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d)|^3) \\ &+ C^{*2}\mathrm{E}(\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d))^4.\end{aligned}$$

But $\mathrm{E}(\frac{1}{d}\xi_d - \mathrm{E}(\frac{1}{d}\xi_d))^2 = \mathrm{D}(\frac{1}{d}\xi_d) = \frac{1-b_d}{4d}$ and

$$\mathrm{E}(\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d))^4 \le 3\frac{(1-b_d)^2}{16d^2} + \frac{1-b_d}{4d^3} = 3\frac{(1-b_d)^2}{16d^2}\left(1 + \frac{4}{(1-b_d)d}\right),$$

because if $S_n = \chi_1 + \cdots + \chi_n$ is a sum of n independent indicators then (it is easy to check by induction)

$$\mathrm{E}(S_n - \mathrm{E}(S_n))^4 = 3(\mathrm{D}(S_n))^2 + \mathrm{D}(S_n) - 6\sum_{k=1}^{n}(\mathrm{D}(\chi_k))^2.$$

Further, according to the Lyapunov inequality and condition $b_d < 1 - \delta$

$$\mathrm{E}(|\tfrac{1}{d}\xi_d - \mathrm{E}\tfrac{1}{d}\xi_d|^3) \le \left(\mathrm{E}(\tfrac{1}{d}\xi_d - \mathrm{E}(\tfrac{1}{d}\xi_d))^4\right)^{3/4} \le 3\frac{(1-b_d)^{3/2}}{8d^{3/2}}\left(1 + \frac{4}{\delta d}\right),$$

so

$$\begin{aligned}\mathrm{D}(s(\tfrac{1}{d}\xi_d)) &\le (s'(\mathrm{E}(\tfrac{1}{d}\xi_d)))^2\frac{1-b_d}{4d} + 3\left(1+\frac{4}{\delta d}\right)\left(2C^*s'(\mathrm{E}(\tfrac{1}{d}\xi_d))\frac{(1-b_d)^{3/2}}{8d^{3/2}} + \frac{(1-b_d)^2}{16d^2}\right) \\ &= \frac{1+\sqrt{1-a_d^2}}{1-a_d^2}(1 - b_d + o(1)) \qquad \text{as } d \to \infty,\end{aligned}$$

and equality (7) and Theorem 4 are proven.

If $X = (x_1, \dots, x_d)$, $X_1, X_2, \dots$ are independent identically distributed random vectors with values in V_d with independent components,

$$\Pr\{x_j = 1\} = \tfrac{1}{2} + \varepsilon_j, \quad \Pr\{x_j = -1\} = \tfrac{1}{2} - \varepsilon_j, \qquad j = 1, \dots, d, \quad \sum_{j=1}^{d} \varepsilon_j^2 > 0,$$

then their distribution is asymmetric, U-statistics

$$T_n = \frac{1}{n\sqrt{d}} \sum_{1 \le i < j \le n} (\|X_i + X_j\| - \|X_i - X_j\|)$$

are nondegenerate and according to Hoeffding (1948) distributions of T_n as $n \to \infty$ are asymptotically normal with parameters

$$\left(\tfrac{n}{2\sqrt{d}} \mathrm{E}(\|X_1 + X_2\| - \|X_1 - X_2\|), \tfrac{4n}{d} \mathrm{E}(D\{\|X_1 + X_2\| - \|X_1 - X_2\| \big| X_1\}) \right).$$

For finite d and fixed $\varepsilon_1, \dots, \varepsilon_d$ the parameters of asymptotic normality take concrete values.

Let the conditions of Theorem 3 be now fulfilled. Then we may use the results of Mihailov (1975) (in this paper the central limit theorem for U-statistics was proven by the method of moments under the assumption that the distributions of X_i and the form of the kernels may depend on n). In this case T_n are asymptotically normal with parameters

$$\left(\frac{n a_d \sqrt{2}}{\sqrt{1 - a_d} + \sqrt{1 + a_d}}, \frac{n(a_d - b_d)}{d} \frac{1 + \sqrt{1 - a_d^2}}{1 - a_d^2} \right).$$

We omit the proofs of these formulas.

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