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The Queueing Model $MAP|PH|1|N$ with Feedback Operating in a Markovian Random Environment

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Abstract: Queueing systems with feedback are well suited for the description of message transmission and manufacturing processes where a repeated service is required. In the present paper we investigate a rather general single server queue with a Markovian Arrival Process (MAP), Phase-type (PH) service-time distribution, a finite buffer and feedback which operates in a random environment. A finite state Markovian random environment affects the parameters of the input and service processes and the feedback probability. The stationary distribution of the queue and of the sojourn times as well as the loss probability are calculated. Moreover, Little's law is derived.

Keywords: Feedback Queue, Batch Markovian Arrival Process, BMAP, Phase-Type Service, PH, Little's Law.

1 Introduction

The traditional assumption in queueing is the following: after the server completes the service of a customer, this customer leaves the system forever and does not affect the further operation of the system. However, in some real systems a quality control is implemented for the service after the service completion and with some probability the customer may return to get additional service. Such a situation takes place,for instance, when a message is transmitted along a noisy wireless channel. These models with a return of a customer to get additional service (feedback queueing models) deserve special consideration. In this respect, we can refer to Takacs' pioneering work (Takacs, 1963).

Compared to the existing literature two main contributions are made in the present paper. At first, we take into account the correlated nature of arrival streams in modern systems, e.g. message flows in telecommunication networks. Therefore, we apply so called Markovian Arrival Processes (MAP) as input of the system. They are used instead of a Poisson process which is one of the most popular input streams in the literature. Moreover, the statistical analysis of measurements in the modern telecommunication networks, for instance, has shown that the stationary Poisson process does not fit well to experimental data.

The second and more important contribution concerns a situation where the parameters of the input and service processes and the feedback probability to get the additional service depend on the current state of some external random process called a random environment. In addition, we deal with a finite buffer while previous studies have mainly considered feedback models with infinite buffer.

2 Mathematical Model

We consider a single server system with a finite buffer of capacity $N-1$, $N \geq 1$. The behavior of the system depends on the state of a stochastic process, i.e. the random environment, ξ_t,$t \geq 0$. It is assumed to be an irreducible continuous-time Markov chain with the state space $\{1, \ldots, M\}$, $M \geq 2$ and the infinitesimal generator Q.

The input flow of the system is determined by the following modification of the well-known BMAP (see, e.g., Lucantoni, 1991; Chakravarthy, 2001). In this input flow, the inter-arrival times of customers are directed by an irreducible continuous time process ν_t, $t \geq 0$ (directing process) with the state space $\{0, 1, \ldots, W\}$. Given a fixed state m of the random environment, this process behaves as follows. The sojourn time of the process ν_t in state ν is exponentially distributed with parameter $\lambda_\nu^{(m)}$, $\lambda_\nu^{(m)} > 0$, $\nu = \overline{0, W}$. After this sojourn time has expired, the process ν_t either jumps to the state r, $r = \overline{0, W}$, $\nu \neq r$, without generating an arrival with probability $p_0^{(m)}(\nu, r)$, or the process ν_t jumps to the state r, $r = \overline{0, W}$ generating a batch arrival of size $k \geq 1$ with probability $p_k^{(m)}(\nu, r)$, $k \geq 1$, $m = \overline{1, M}$,

$$\sum_{r=0, r\neq\nu}^{W} p_0^{(m)}(\nu, r) + \sum_{k=1}^{\infty} \sum_{r=0}^{W} p_k^{(m)}(\nu, r) = 1, \nu = \overline{0, W}, m = \overline{1, M}.$$

We introduce the matrices $D_k^{(m)}$, $k \geq 0$, $m = \overline{1, M}$, whose elements are defined by:

$$\begin{aligned}
(D_0^{(m)})_{\nu,\nu} &= \lambda_\nu^{(m)}, \quad \nu = \overline{0, W}, \\
(D_0^{(m)})_{\nu,r} &= \lambda_\nu^{(m)} p_0^{(m)}(\nu, r), \quad \nu, r = \overline{0, W}, \quad \nu \neq r, \\
(D_k^{(m)})_{\nu,r} &= \lambda_\nu^{(m)} p_k^{(m)}(\nu, r), \quad k \geq 1, \nu, r = \overline{0, W}.
\end{aligned}$$

Then, the MAP input flow is completely defined by the set of only two types of matrices $D_k^{(m)}$, $k = 0, 1$, $m = \overline{1, M}$.

A customer who meets a full buffer is rejected and lost.

The service process is defined by the modification of the Phase-type service-time distribution (see, e.g., Neuts, 1981). The service time is interpreted as the first passage time until a continuous-time Markov chain η_t, $t \geq 0$ reaches the absorbing state. This chain has the state space $\{1, \ldots, K\}$. Given a fixed value m of the random environment ξ_t, $t \geq 0$, transitions of the chain η_t, $t \geq 0$ within the state space are defined by the irreducible sub-generator $S^{(m)}$ while the intensities of transitions into the absorbing state are defined by the vector $\mathbf{S}_0^{(m)} = -S^{(m)}\mathbf{e}$. Here $\mathbf{e}$ is a column vector of all ones of appropriate size. At the beginning of the service, the state of the process η_t, $t \geq 0$ is chosen according to the row vector $\boldsymbol{\beta}^{(m)}$ of probabilities, $m = \overline{1, M}$.

At service completion, the customer who got the service leaves the system forever with probability p_m when the current state of the random environment is m, $m = \overline{1, M}$. With the complementary probability $1 - p_m$, the customer immediately returns to the server to get repeated processing.

The admitted customers are served according to a First In - First Out (FIFO) discipline.

3 Stationary Distribution of the Queue Length

We consider the four-dimensional continuous-time process $\zeta_t = \{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$, where i_t is the number of customers in the system at the epoch t, $t \geq 0$, $i_t = i, i = \overline{0,N}$, the components $\{\xi_t, \nu_t, \eta_t\}$, $t \geq 0$ are defined above, $\xi_t = m, m = \overline{1,M}$, $\nu_t = \nu, \nu = \overline{0,W}$, $\eta_t = n, n = \overline{1,K}$. It is easy to see that this process is a Markov chain. Due to the assumptions made about the random environment process ξ_t, $t \geq 0$, the underlying process ν_t, $t \geq 0$, governing the MAP and the process η_t, $t \geq 0$, the process ζ_t, $t \geq 0$ is a regular irreducible continuous-time Markov chain with a finite state space. Due to Foster's criterion, it has a unique stationary distribution. We denote its stationary state probabilities by

$$\begin{aligned} p(0,m,\nu) &= \lim_{t\to\infty} P\{i_t = 0, \xi_t = m, \nu_t = \nu\} \\ p(i,m,\nu,n) &= \lim_{t\to\infty} P\{i_t = i, \xi_t = m, \nu_t = \nu, \eta_t = n\}, \\ &\quad i = \overline{1,N}, m = \overline{1,M}, \nu = \overline{0,W}, n = \overline{1,K}. \end{aligned}$$

Enumerating the states of the chain ζ_t, $t \geq 0$ in lexicographical order, $\mathbf{p}_i$, $i = \overline{0,N}$ represents the vector of probabilities corresponding to the state i of the entry i_t. Let $\mathbf{p} = (\mathbf{p}_0, \ldots, \mathbf{p}_N)$.

Lemma 1. The vectors $\mathbf{p}_i$, $i = \overline{0,N}$, satisfy the system

$$\begin{array}{l} \mathbf{p}_0\widetilde{\mathcal{C}} + \mathbf{p}_1\widetilde{\mathcal{H}} = \mathbf{0}, \\ \mathbf{p}_1\mathcal{C} + \mathbf{p}_0\widetilde{\mathcal{D}_1} + \mathbf{p}_2\mathcal{H} = \mathbf{0}, \\ \mathbf{p}_i\mathcal{C} + \mathbf{p}_{i-1}\mathcal{D}_1 + \mathbf{p}_{i+1}\mathcal{H} = \mathbf{0}, \quad i = \overline{2,N-1}, \\ \mathbf{p}_N\left(\mathcal{C} + \mathcal{D}_1\right) + \mathbf{p}_{N-1}\mathcal{D}_1 = \mathbf{0}, \end{array} \tag{1}$$

where

$$\begin{array}{l} \mathcal{D}_k = diag\{D_k^{(m)} \otimes I_K,\ m = \overline{1,M}\}, \quad k = 0,1, \\ \widetilde{\mathcal{D}}_1 = diag\{D_1^{(m)} \otimes \boldsymbol{\beta}^{(m)},\ m = \overline{1,M}\}, \\ \widetilde{\widetilde{\mathcal{D}}}_k = diag\{D_k^{(m)}, \quad m = \overline{1,M}\}, \quad k = 0,1, \\ C = Q \otimes I_{\overline{W}K} + \mathcal{D}_0 + \mathcal{S} - \mathcal{H}, \\ \widetilde{C} = Q \otimes I_{\overline{W}} + \widetilde{\widetilde{\mathcal{D}}}_0, \\ \mathcal{H} = diag\{I_{\overline{W}} \otimes \mathbf{S}_0^{(m)}\boldsymbol{\beta}^{(m)}p_m,\ m = \overline{1,M}\}, \\ \widetilde{\mathcal{H}} = diag\{I_{\overline{W}} \otimes \mathbf{S}_0^{(m)}p_m,\ m = \overline{1,M}\}, \\ \mathcal{S} = diag\{I_{\overline{W}} \otimes (S^{(m)} + \mathbf{S}_0^{(m)}\boldsymbol{\beta}^{(m)}),\ m = \overline{1,M}\}. \end{array}$$

Here $diag\{\}$ represents the diagonal matrix with the diagonal entries defined in brackets, $\otimes$ is the Kronecker product of matrices, **0** is a row vector of zeros, I is the identity matrix of a dimension defined by the context or by the suffix $\overline{W} = W + 1$.

Proof. The proof of the lemma is implemented by a standard calculation of the transition probabilities of the Markov chain. during a very small time interval. ■

Theorem 1. The vectors $\mathbf{p}_i$, $i = \overline{0,N}$, of the stationary probabilities are calculated as follows:

$$\mathbf{p}_i = \mathbf{p}_0 \Phi_i, \quad i = \overline{1, N},$$

where $\mathbf{p}_0$ is the unique solution of the following system of linear equations:

$$\mathbf{p}_0 \left(\widetilde{\mathcal{C}} + G_1 \widetilde{\mathcal{H}} \right) = \mathbf{0}, \qquad \mathbf{p}_0 \sum_{i=0}^{N} \Phi_i \mathbf{e} = 1,$$

$$\Phi_0 = I, \qquad \Phi_i = \Phi_{i-1} G_i, \qquad i = \overline{1, N}.$$

The matrices G_i are defined as follows:

- for $N = 1$,

$$G_1 = -\widetilde{\mathcal{D}}_1 \left(\mathcal{C} + \mathcal{D}_1 \right)^{-1},$$

- for $N > 1$,

$$G_i = \begin{cases} -\widetilde{\mathcal{D}}_1 \left(\mathcal{C} + G_2 \mathcal{H} \right)^{-1}, & i = 1, \\ -\mathcal{D}_1 \left(\mathcal{C} + G_{i+1} \mathcal{H} \right)^{-1}, & i = \overline{2, N-1}, \\ -\mathcal{D}_1 \left(\mathcal{C} + \mathcal{D}_1 \right)^{-1}, & i = N. \end{cases}$$

Proof. The proof follows from the application of the reduction algorithm to the system (1) starting at the last equation. ■

4 Sojourn Time Distribution

From a practical perspective, the sojourn time of a customer and its mean value are some of the most important performance measures of any queueing system. For a finite queueing system, the following two versions of the sojourn time are considered: (i) the sojourn time for an arbitrary customer, (ii) the sojourn time for a successful customer, i.e. a customer that is not lost due to a full buffer at its arrival. In this paper, both versions are considered.

First of all, we consider the average intensity of the input flow. It is defined by

$$\lambda = \mathbf{x} \, \widetilde{\widetilde{\mathcal{D}}}_1 \, \mathbf{e}, \tag{2}$$

where $\mathbf{x}$ is the unique solution of the following system of linear equations:

$$\mathbf{x} \left(\widetilde{\mathcal{C}} + \widetilde{\widetilde{\mathcal{D}}}_1 \right) = \mathbf{0}, \quad \mathbf{x}\mathbf{e} = 1. \tag{3}$$

Lemma 2. The average intensity of the input flow λ is calculated as follows:

$$\lambda = \mathbf{p}_0 \widetilde{\mathcal{D}}_1 \mathbf{e} + \sum_{i=1}^{N} \mathbf{p}_i \mathcal{D}_1 \mathbf{e}. \tag{4}$$

Proof. Multiplying both sides of system (1) by $I_{M(W+1)} \otimes \mathbf{e}_K$ and summing them up, we get the following system of linear equations:

$$\left(\mathbf{p}_0 + \sum_{i=1}^{N} \mathbf{p}_i \left(I_{M(W+1)} \otimes \mathbf{e}_K \right) \right) \left(\widetilde{\mathcal{C}} + \widetilde{\widetilde{\mathcal{D}}}_1 \right) = \mathbf{0}. \tag{5}$$

Taking into account the normalization condition $\sum_{i=0}^{N} \mathbf{p}_i \mathbf{e} = 1$ and the expression above, it can be easily seen that $\mathbf{x} = \mathbf{p}_0 + \sum_{i=1}^{N} \mathbf{p}_i \left(I_{M(W+1)} \otimes \mathbf{e}_K \right)$. In this way, the expression (2) is equivalent to (4). ■

Secondly, we have to calculate the value of the probability P_{loss} that an arbitrary customer is lost in front of the system.

Lemma 3. The loss probability P_{loss} is calculated as follows:

$$P_{loss} = \lambda^{-1} \mathbf{p}_N \mathcal{D}_1 \mathbf{e}. \tag{6}$$

Proof. It is obvious that the customer is lost if there are N customers in the system at his arrival epoch. The term $\lambda^{-1} \mathbf{p}_N \mathcal{D}_1$ defines the distribution of the finite components $\{\xi_t, \nu_t, \mu_t\}$ of the Markov chain ζ_t, $t \geq 0$ just after such an arrival epoch. ■

At third, we have to find the joint stationary distribution $\boldsymbol{\pi}_i$,$i = \overline{0, N}$ of the processes i_t, ξ_t, η_t at the epoch just before the arrival of a customer and of the process $\nu_t, t \geq 0$ after this arrival epoch.

Theorem 2. The vector generating function $\mathbf{\Pi}(z) = \sum_{i=0}^{N} \boldsymbol{\pi}_i z^i$, $|z| \leq 1$ satisfies the following equation:

$$z\mathbf{\Pi}(z) = z^{N+1} P_{loss} + \lambda^{-1} \sum_{i=1}^{N} \mathbf{p}_i \left(\mathcal{H} - \sum_{j=0}^{i-1} z^j \left(\Gamma z + \mathcal{H} \right) \right), \tag{7}$$

where $\Gamma = \mathcal{C} + \mathcal{D}_1$.

Proof. It can be shown that $\boldsymbol{\pi}_0 = \lambda^{-1} \mathbf{p}_0 \widetilde{\mathcal{D}_1}$, $\boldsymbol{\pi}_i = \lambda^{-1} \mathbf{p}_i \mathcal{D}_1$, $i = \overline{1, N}$. Multiplying both sides of the equilibrium equations (1) by z^i,$i = \overline{1, N}$ and summing them up, we get the following equation for the vector generating function $\mathbf{p}(z) = \sum_{i=1}^{N} \mathbf{p}_i z^i$:

$$\begin{aligned} &\mathbf{p}(z)\mathcal{C} + \mathbf{p}_N \mathcal{D}_1 z^N \\ &+ \left(\mathbf{p}(z) z - \mathbf{p}_N z^{N+1} \right) \mathcal{D}_1 + z \mathbf{p}_0 \widetilde{\mathcal{D}_1} + z^{-1} \left(\mathbf{p}(z) - z \mathbf{p}_1 \right) \mathcal{H} = \mathbf{0}. \end{aligned} \tag{8}$$

We rewrite (8) in the form:

$$\begin{aligned} &(z-1) \left(z \mathbf{p}_0 \widetilde{\mathcal{D}_1} + z \mathbf{p}(z) \mathcal{D}_1 \right) \\ &= z \left(\mathbf{p}_1 \mathcal{H} - \mathbf{p}_0 \widetilde{\mathcal{D}_1} \right) + (z-1) z^{N+1} \mathbf{p}_N \mathcal{D}_1 - \mathbf{p}(z) \left(\Gamma z + \mathcal{H} \right). \end{aligned} \tag{9}$$

Substituting $z = 1$ into (9), we get:

$$\mathbf{p}(1) \left(\Gamma + \mathcal{H} \right) = \mathbf{p}_1 \mathcal{H} - \mathbf{p}_0 \widetilde{\mathcal{D}_1}. \tag{10}$$

We substitute $\left(\mathbf{p}_1 \mathcal{H} - \mathbf{p}_0 \widetilde{\mathcal{D}_1} \right)$ in (9) using (10) and get:

$$\begin{aligned} &(z-1) \left(z \mathbf{p}_0 \widetilde{\mathcal{D}_1} + z \mathbf{p}(z) \mathcal{D}_1 \right) \\ &= (z-1) z^{N+1} \mathbf{p}_N \mathcal{D}_1 + (z-1) \mathbf{p}(1) \mathcal{H} - (z-1) \sum_{i=1}^{N} \mathbf{p}_i \sum_{j=0}^{i-1} z^j \left(\Gamma z + \mathcal{H} \right) \end{aligned} \tag{11}$$

Dividing both sides of (11) by $\lambda(z-1)$, we derive (7). ■

Theorem 3. The Laplace-Stieltjes transform of the sojourn-time distribution of an arbitrary customer has the following form:

$$W(s) = \lambda^{-1} \sum_{i=1}^{N} \mathbf{p}_i \left(\mathcal{H}\mathbf{e} - s \sum_{j=1}^{i} (\mathcal{Z}(s))^j \mathbf{e} \right) + P_{loss}, \quad s \in \not{C}, Res > 0 \tag{12}$$

where $\mathcal{Z}(s) = (sI - \Gamma)^{-1}\mathcal{H}, \quad s \in \not{C}, Re\ s > 0$.

Proof. The proof is based on the probabilistic interpretation of the Laplace-Stieltjes transform.

We assume that, independently of the system operation, a stationary Poissonian input of so called catastrophes arrives. Let $s > 0$ be the intensity of this flow.

Let $\widetilde{W}(x)$ be the distribution function of the sojourn time of an arbitrary customer. Then, its Laplace-Stieltjes transform

$$W(s) \stackrel{def}{=} \int_0^\infty e^{-sx} d\widetilde{W}(x)$$

can be interpreted as the probability of no arrivals of catastrophes during the sojourn time. This argument allows us to derive the expression for $W(s)$ by probabilistic reasoning. The obtained results are valid only for real $s > 0$. But there exists the unique analytic continuation of the obtained function to the right half of the complex plane $s \in \not{C}, Re\ s > 0$.

It is easy to see that the function $W(s)$ can be calculated by the formula of total probability in the following form:

$$W(s) = \boldsymbol{\pi}_0 \mathbf{W}_1(s) + \sum_{i=1}^{N-1} \boldsymbol{\pi}_i \mathbf{W}_{i+1}(s) + \boldsymbol{\pi}_N \mathbf{e}. \tag{13}$$

The entries $(\mathbf{W}_i(s))_{m,\nu,\eta}$ of the vector $\mathbf{W}_i(s)$ are the probabilities of no arrivals of catastrophes during the virtual sojourn time with i customers at the system and a given state (m, ν, η) of the process $\{\xi_t, \nu_t, \eta_t\}$, $t \geq 0$ at the arrival epoch of the virtual customer. The vector $\mathbf{W}_i(s)$ is calculated by

$$\mathbf{W}_i(s) = \int_0^\infty e^{-st}\, P_i(t, 0)\, \mathcal{H}\, dt\, \mathbf{e}, \tag{14}$$

where the matrices $P_i(t, l)$ have entries $(P_i(t, l))_{m,\nu,\eta;m',\nu',\eta'}$. The latter are defined as the probability to have l customers in the system in front of the tagged customer and to observe the state (m', ν', η') of the process $\{\xi_\tau, \nu_\tau, \eta_\tau\}$, $\tau \geq 0$ at the epoch t provided that i, $0 \leq l \leq i$, customers were in front of this customer and the state of the process $(\xi_\tau, \nu_\tau, \eta_\tau)$, $\tau \geq 0$, was given by (m, ν, η) at the epoch 0.

We combine the matrices $P_i(t, l)$ into the block row vector:

$$\mathcal{P}_i(t) = (P_i(t, 0), \ldots, P_i(t, i))$$

Then the differential equation

$$\frac{d}{dt}\mathcal{P}_i(t) = \mathcal{P}_i(t)\Psi_i$$

can be derived. Here all blocks of the matrix Ψ_i except the diagonal blocks which are equal to Γ and the sub-diagonal blocks, that are equal to $\mathcal{H}$, are zero. Solving this equation with an obvious initial condition

$$\mathcal{P}_i(0) = (0, ..., 0, I)$$

and exploiting (14), we get the formula

$$\mathbf{W}_i(s) = (0, ..., 0, I)(sI - \Psi_i)^{-1}(I, 0, ..., 0)^T\mathcal{H}\mathbf{e} = (\mathcal{Z}(s))^i\,\mathbf{e}.$$

Note that the matrix $\mathcal{H}$ should be replaced by the matrix $\widetilde{\mathcal{H}}$ in the case when the tagged customer is the single one in the system at the moment of his departure. But $\mathcal{H}\mathbf{e} = \widetilde{\mathcal{H}}\mathbf{e}$ and there is no need to consider this case separately.

The determinant of the matrix $(sI - \Gamma)$ is not equal to zero for $Re\ s \geq 0$ due to O. Tausska's theorem, see Gantmacher (1967).

At this stage, the expression of $W(s)$ is as follows:

$$W(s) = \lambda^{-1}\left(\mathbf{p}_0\widetilde{\mathcal{D}_1}\mathcal{Z}(s)\mathbf{e} + \sum_{i=1}^{N-1}\mathbf{p}_i\mathcal{D}_1\left(\mathcal{Z}(s)\right)^{i+1}\mathbf{e}\right) + P_{loss}, \quad Re\ s \geq 0.$$

Substituting z in (7) by $\mathcal{Z}(s)$ and noting that $\Gamma\mathcal{Z}(s) + \mathcal{H} = s\mathcal{Z}(s)$, we finally derive (12). ■

Theorem 4. The Laplace-Stieltjes transform $W_{success}(s)$ of the sojourn-time distribution of a successful customer has the following form:

$$W_{success}(s) = \frac{\sum\limits_{i=1}^{N}\mathbf{p}_i\left(\mathcal{H}\mathbf{e} - s\sum\limits_{j=1}^{i}\left(\mathcal{Z}(s)\right)^j\mathbf{e}\right)}{\lambda(1 - P_{loss})}, \quad Re\ s > 0.$$

Proof. Regarding $W_{success}(s)$ the formula of total probability has the following form:

$$W_{success}(s) = \boldsymbol{\alpha}_0\mathbf{W}_1(s) + \sum_{i=1}^{N-1}\boldsymbol{\alpha}_i\mathbf{W}_{i+1}(s).$$

Here $\boldsymbol{\alpha}_i, i = \overline{0, N-1}$ are the stationary probabilities of the components $\{i_t, \xi_t, \eta_t\}, t \geq 0$ of the process $\zeta_t, t \geq 0$ at the epoch before the arrival of a successful customer and of the process $\nu_t, t \geq 0$ after this arrival epoch. It can be shown that $\boldsymbol{\alpha}_0 = (\lambda(1-P_{loss}))^{-1}\mathbf{p}_0\widetilde{\mathcal{D}_1}$, $\boldsymbol{\alpha}_i = (\lambda(1-P_{loss}))^{-1}\mathbf{p}_i\mathcal{D}_1$, $i = \overline{1, N-1}$. Using this fact, we can easily repeat the proof of the previous Theorem for our case. ■

Corollary. The mean sojourn times W and $W_{success}$ of an arbitrary and a successful customer, respectively, satisfy Little's law:

$$\lambda\, W = L,$$
$$\lambda(1 - P_{loss})\, W_{success} = L,$$

where $L = \sum\limits_{i=1}^{N} i\mathbf{p}_i\mathbf{e}$ is the mean number of customers in the system.

5 Numerical Examples

To illustrate the feasibility and outcome of the presented algorithms, we consider the following example.

Let the Markov chain $\xi_t,\ t \geq 0$ of the random environment have two states. It is defined by the infinitesimal generator:

$$Q = \begin{pmatrix} -1.4 & 1.4 \\ 1.8 & -1.8 \end{pmatrix}.$$

The MAP input is characterized by the matrices

$$D_0^{(1)} = \begin{pmatrix} -86 & 0.01 \\ 0.02 & -2.76 \end{pmatrix}, \quad D_1^{(1)} = \begin{pmatrix} 85 & 0.99 \\ 0.2 & 2.54 \end{pmatrix}$$

when the random environment stays in the state 1 and by the matrices

$$D_0^{(2)} = \begin{pmatrix} -8 & 1 \\ 2 & -12 \end{pmatrix}, \quad D_1^{(2)} = \begin{pmatrix} 2 & 5 \\ 4 & 6 \end{pmatrix}$$

when the state of the random environment is determined by 2.

The service-time distribution of Phase-type is characterized by the vectors $\boldsymbol{\beta}^{(1)} = (0.2, 0.8)$, $\boldsymbol{\beta}^{(2)} = (0.9, 0.1)$ and the sub-generators

$$S^{(1)} = \begin{pmatrix} -170 & 15 \\ 40 & -210 \end{pmatrix}, \quad S^{(2)} = \begin{pmatrix} -110 & 80 \\ 10 & -150 \end{pmatrix}.$$

In the experiment, we fix the parameters of the MAP input and PH services as well as the queue capacity $N = 10$ and change the probabilities p_m to leave the system after the service completion when the random environment is in the state $m, m = 1, 2$.

The Figures 1,2,3 illustrate the dependence of the values L, $\mathbf{p}_0\mathbf{e}$, P_{loss} on the probabilities p_1 and p_2. As noticed by these figures, the observed values change in a monotone manner until the point $p_1 = p_2 = 1$, when the system is completely unavailable for incoming customers. Here $L = N$, $\mathbf{p}_0\mathbf{e} = 0$, $P_{loss} = 1$ hold.

6 Conclusion

The feedback queueing model of the type $MAP|PH|1|N$ operating in a Markovian random environment has been investigated. The random environment has a finite state space. Changes of its state causes instantaneous changes of the parameters of the MAP input and the Phase-type service processes as well as the probability of a repeated service.

The stationary distribution of the associated multi-dimensional continuous-time Markov chain describing the behavior of the system has been calculated. Furthermore, the Laplace-Stieltjes transform of the sojourn-time distribution of an arbitrary customer and a successful customer and a variant of Little's law has been derived.

The work of the derived elaborated algorithms has been illustrated by numerical examples.

The presented results can be applied to the capacity planning of realistic feedback systems and the performance evaluation in situations where a repeated service of objects is required and the operation on the object is subject to some external influence.

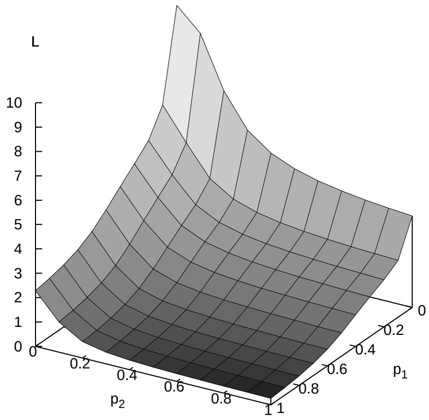

Figure 1: Dependence of the average queue length L on the feedback probabilities p_1 and p_2

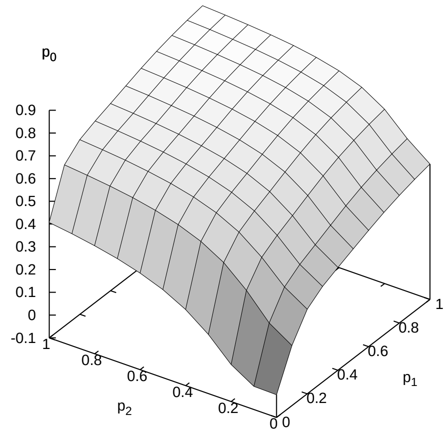

Figure 2: Dependence of the probability of an idle state $\mathbf{p}_0\mathbf{e}$ on the feedback probabilities p_1 and p_2

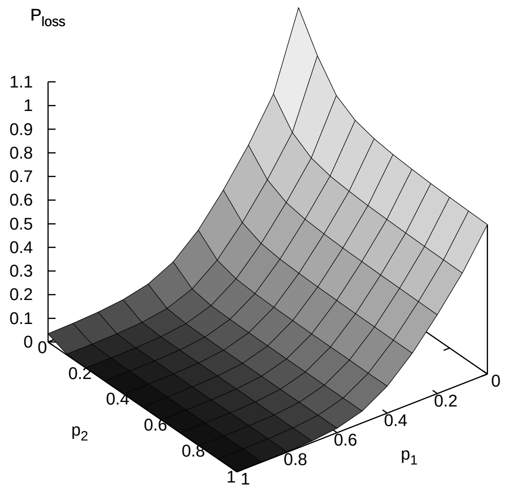

Figure 3: Dependence of the loss probability P_{loss} on the feedback probabilities p_1 and p_2

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