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Shrinkage Estimators for the Exponential Scale Parameter under Multiply Type II Censoring

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Abstract: We consider the problem of estimating the scale parameter of an exponential distribution under multiply type II censoring when a prior point guess of the parameter value is available. Shrinkage estimators are obtained from the approximate maximum likelihood estimators proposed in Singh et al. (2004) and in Balasubramanian and Balakrishnan (1992). These estimators are then compared by their simulated mean squared errors.

Zusammenfassung: Wir behandeln das Problenm der Schätzung des Skalenparameters einer Exponentialverteilung unter mehrfacher Typ II Zensierung der Stichprobe. Dazu nehmen wir an, dass bereits vorweg auch eine Vermutung über den Wert des Parameters vorliegt. Wir erhalten Shrinkage-Schätzer aus den approximativen Maximum-Likelihood Schätzern in Singh et al. (2004) und Balasubramanian and Balakrishnan (1992). Ein Vergleich dieser Schätzer beruht dann auf deren simulierten mittleren quadratischen Fehler.

Keywords: Exponential Distribution, Approximate Maximum Likelihood Estimator, Prior Point Guess, Monte Carlo Simulation.

1 Introduction

In life testing experiments a fixed number of items, say n, is often put on test simultaneously. But the experimenter may not always be in a position to observe the life times of all these items because of time limitations or other restrictions on the data collection process. Let us suppose that out of the n items only the first l life-times have been observed and the life-times of the other $(n-l)$ components remain unobserved and are missing. This type of censoring is known as right type II censoring. Another way to get censored data is to observe only the largest m life times. In this case the life times of the first $(n-m)$ components are missing. Such censoring is known as a left type II censoring scheme (see Leemis and Shih, 1989). Moreover, if left and right censoring happen together, this is known as doubly type II censoring (see Sarhan and Greenberg, 1957). A reverse situation to doubly type II censoring is mid censoring, where the data on two extremes are available but some of the middle observations are censored (see Sarhan and Greenberg, 1962). If mid censoring arises amongst doubly censored observations, the scheme is known as a multiply type II censoring scheme. Balakrishnan (1990) has discussed a more general version of such a multiply type II censoring, where only the r_1th, r_2th, ..., r_kth $(1 \leq r_1 < \cdots < r_k \leq n)$ failure times are available. Under this multiply type II censoring scheme even the likelihood estimate for the one parameter exponential distribution is difficult to obtain directly from the likelihood equation. Balasubramanian and Balakrishnan (1992) and Singh et al. (2004) proposed some approximate maximum likelihood estimators, which are denoted as $\hat{\theta}_{BL}$ and $\hat{\theta}_{UA}$, respectively.

In real world situations, particularly in life testing problems, the experimenter may have evidence that the value of the parameter under study, say θ, is in the neighborhood of θ_0. We call θ_0 the experimenter's prior point guess. For example, for a patient suffering from cancer the doctor may believe that the patient will survive two more months. In this case θ_0 can be taken to be equal to two months. Similarly, a bulb producer may know that the average life time of his product may be close to 900 hours. Here we may take $\theta_0 = 900$. Now the following questions arise: "Should we use θ_0 in the estimation procedure, which may be a close guess of θ but may not be its true value?", or "Should we base our estimator on sample information only?" Furthermore, if one wishes to incorporate the additional information θ_0 in the estimation of θ the question may be "How to use it?"

The purpose of this paper is to study the procedures, which answer the above questions in order to estimate the scale parameter of an exponential distribution under a multiply type II censoring scheme. It may be recalled that Thompson (1968) was the first who proposed a procedure popularly known as shrinkage procedure, which suggests the use of a prior point guess of the parameter for improving the performance of the existing estimator $\hat{\theta}$. If a prior point guess θ_0 is available with known confidence α, $0 < \alpha < 1$, the shrinkage estimator for θ is defined as

$$T_{SH} = \alpha\theta_0 + (1-\alpha)\hat{\theta}\,. \tag{1}$$

Using Thompson's technique, the respective shrinkage estimators based on the approximate maximum likelihood estimators $\hat{\theta}_{UA}$ and $\hat{\theta}_{BL}$ can easily be defined. Studies of such types of other estimators reveal that these perform better than the original estimators provided the true value of θ is close to θ_0 and α is taken to be large. It is also noted that the performance of these estimators strongly depends on the choice of α. If α is not set in accordance with the reality (i.e., large α when θ is close to θ_0, and small α when θ is away from θ_0), it may happen that either there is no significant gain in the performance of T_{SH} or there is actually a significant loss. In general, the true value of the parameter is unknown and, hence, a proper choice of α can not be guaranteed. Therefore, in the situations when the experimenter is either not able to provide a fixed value of α or it is feared that the value of α may not be in accordance with the real situation, it may be proposed to consider (1) as a class of estimators and select the best by choosing α such that the mean squared error (MSE) of T_{SH} is at its minimum. It is easy to verify that the optimum value of α for which $\mathrm{MSE}(T_{SH})$ is minimized, is

$$\alpha_{opt} = \frac{\mathrm{MSE}^2(\hat{\theta}) - (\theta_0-\theta)\mathrm{bias}(\hat{\theta})}{(\theta_0-\theta)^2 + \mathrm{MSE}(\hat{\theta}) - 2(\theta_0-\theta)\mathrm{bias}(\hat{\theta})}\,. \tag{2}$$

It is clear that α_{opt} depends on θ. It is therefore suggested to replace θ in (2) by its estimate, giving $\hat{\alpha}_{opt}$. Needless to mention that due to the use of $\hat{\theta}$ in α_{opt}, the performance of the shrinkage estimator is expected to be adversely affected.

Comparisons of the performance of shrinkage estimators with the usual estimators are quite common in the existing literature. But the present paper discusses for the first time the effect of the use of different estimators on the corresponding shrinkage estimators. Comparing MSEs, it will be seen that shrinkage estimators based on the approximate likelihood estimator proposed by Singh et al. (2004) perform better than the one based on results in Balasubramanian and Balakrishnan (1992). Singh et al. (2004) proposed a procedure to obtain an approximate maximum likelihood estimator as an alternative to the one given in Balasubramanian and Balakrishnan (1992). The present paper aims to develop the shrinkage estimators from these approximate maximum likelihood estimators and compare their performances.

In the next section we obtain the shrinkage estimators for θ using a prior point guess. In Section 3 the proposed estimators are computed for the data given in Balasubramanian and Balakrishnan (1992) in order to illustrate the procedure discussed here. The MSEs of all estimators are then compared in Section 5. Finally, a brief conclusion is given.

2 Shrinkage Estimation

Consider a one parameter exponential distribution with pdf

$$f(x|\theta) = \frac{1}{\theta}\exp(-x/\theta)\,, \qquad x \geq 0\,,\ \theta > 0\,. \tag{3}$$

Suppose that n items, whose life-times follow model (3), are placed on test and that the r_1th, r_2th, …, r_kth failure times are recorded as $x_1, \dots, x_k$, respectively. The likelihood function for such a multiply type II censored sample is

$$L(\theta|x) = \frac{n!\,\theta^{-k}\left(1 - e^{-x_1/\theta}\right)^{r_1 - 1}}{(n-r_k)!\,(r_1-1)!\,\prod_{i=1}^{k-1} u_i!}\prod_{i=1}^{k-1}\left(e^{-x_i/\theta} - e^{-x_{i+1}/\theta}\right)^{u_i} e^{-t_k/\theta}\,,$$

where $u_i = r_{i+1} - r_i - 1$, $i = 1, \dots, k-1$, and $t_k = \sum_{i=1}^{k} x_i + (n - r_k)x_k$.

The approximate likelihood estimator of θ proposed by Singh et al. (2004) is

$$\hat{\theta}_{UA} = \frac{t_k + \sum_{i=1}^{k-1} x_i u_i}{k + (r_1 - 1) + \sum_{i=1}^{k-1} u_i}\,,$$

whereas the one proposed by Balasubramanian and Balakrishnan (1992) is

$$\hat{\theta}_{BL} = \frac{\sum_{i=0}^{k-1}(\delta_i x_i + (1-\delta_i)x_{i+1})u_i + t_k}{k - \sum_{i=0}^{k-1} u_i\gamma_i}\,, \tag{4}$$

with $r_0 = x_0 = 0$, $q_i = 1 - r_i/(n+1)$, $\delta_i = q_i/(q_i - q_{i+1}) - q_i q_{i+1}/(q_i - q_{i+1})^2 \log(q_i/q_{i+1})$, and $\gamma_i = (q_{i+1}\log q_{i+1} - q_i \log q_i)/(q_i - q_{i+1}) + \delta_i \log q_i + (1-\delta_i)\log q_{i+1}$.

2.1 Specified Confidence

As discussed earlier, the shrinkage estimators $\hat{\theta}_{UA(\alpha)}$ or $\hat{\theta}_{BL(\alpha)}$ can be defined by replacing $\hat{\theta}$ in (1) by $\hat{\theta}_{UA}$ or $\hat{\theta}_{BL}$. Their MSEs can be easily obtained from

$$\mathrm{MSE}(\hat{\theta}) = \alpha^2(\theta_0 - \theta)^2 + 2\alpha(1-\alpha)(\theta_0 - \theta)\mathrm{bias}(\hat{\theta}) + (1-\alpha)^2\mathrm{MSE}(\hat{\theta})$$

after replacing $\hat\theta$ by $\hat\theta_{UA}$ and $\hat\theta_{BL}$, respectively. Expressions for $\mathrm{bias}(\hat\theta_{UA})$, $\mathrm{bias}(\hat\theta_{BL})$, $\mathrm{MSE}(\hat\theta_{UA})$, and $\mathrm{MSE}(\hat\theta_{BL})$ can also be easily obtained by using

$$\mathrm{E}(x_i) = \theta S_1(n-r_i, n)\,, \quad \text{and} \quad \mathrm{cov}(x_i, x_j) = \theta^2 S_2(n-r_i, n)\,,\ i \le j\,,$$

where $S_1(a,b) = \sum_{l=a+1}^{b} 1/l$ and $S_2(a,b) = \sum_{l=a+1}^{b} 1/l^2$. With these results we get

$$\begin{aligned}
\mathrm{bias}(\hat\theta_{UA}) &= \frac{\theta}{W}\sum_{i=1}^{k}\beta_i S_1(n-r_i,n)\\
\mathrm{bias}(\hat\theta_{BL}) &= \frac{\theta}{V}\sum_{i=1}^{k}\lambda_i S_1(n-r_i,n)\\
\mathrm{MSE}(\hat\theta_{UA}) &= \frac{\theta^2}{W^2}\left\{\sum_{i=1}^{k-1}\beta_i^2\Big(S_2(n-r_i,n)+S_1^2(n-r_i,n)\Big)\right.\\
&\quad +2\sum_{i<j}^{k-1}\sum\beta_i\beta_j\Big(S_2(n-r_i,n)+S_1(n-r_i,n)S_1(n-r_j,n)\Big)\\
&\quad +\beta_k^2\Big(S_2(n-r_k,n)+S_1^2(n-r_k,n)\Big)\\
&\quad +2\beta_k\sum_{i=1}^{k-1}\beta_i\Big(S_2(n-r_i,n)+S_1(n-r_i,n)S_1(n-r_k,n)\Big)\\
&\quad \left.+\,W^2-2W\left(\sum_{i=1}^{k-1}\beta_i S_1(n-r_i,n)+\beta_k S_1(n-r_k,n)\right)\right\}\\
\mathrm{MSE}(\hat\theta_{BL}) &= \frac{\theta^2}{V^2}\left\{\sum_{i=1}^{k-1}\lambda_i^2\Big(S_2(n-r_i,n)+S_1^2(n-r_i,n)\Big)\right.\\
&\quad +\ 2\sum_{i<j}^{k-1}\sum\lambda_i\lambda_j\Big(S_2(n-r_i,n)+S_1(n-r_i,n)S_1(n-r_j,n)\Big)\\
&\quad +\ \lambda_k^2\Big(S_2(n-r_k,n)+S_1^2(n-r_k,n)\Big)\\
&\quad +\ 2\lambda_k\sum_{i=1}^{k-1}\lambda_i\Big(S_2(n-r_i,n)+S_1(n-r_i,n)S_1(n-r_k,n)\Big)\\
&\quad \left.+\ V^2-2V\left(\sum_{i=1}^{k-1}\lambda_i S_1(n-r_i,n)+\lambda_k S_1(n-r_k,n)\right)\right\}
\end{aligned}$$

with $\beta_i = u_i + 1$ and $\lambda_i = -\beta_i + u_i\delta_i - u_{i-1}\delta_{i-1}$, $i = 1, \ldots, k-1$, but $\beta_k = n - r_k + 1$ and $\lambda_k = n - r_{k-1} - u_{k-1}\delta_{k-1}$, $W = k + r_1 - 1 + \sum_{i=1}^{k-1} u_i$, $V = k - \sum_{i=1}^{k-1} u_i\gamma_i$. The terms δ_i and γ_i, $i = 1, \ldots, k$ have been already defined.

2.2 Unspecified Confidence

As suggested in Section 1, the shrinkage estimator based on $\hat{\theta}_{UA}$ when point guess θ_0 is available with unspecified confidence can be obtained as

$$\hat{\theta}_{UA(\alpha)} = \alpha_{UA}\theta_0 + (1-\alpha_{UA})\hat{\theta}_{UA}\,,$$

where α_{UA}, as defined in (2), can be rewritten as

$$\alpha_{UA} = \frac{\mathrm{MSE}(\hat{\theta}_{UA})/\theta^2 - (\theta_0/\theta - 1)\left(\mathrm{bias}(\hat{\theta}_{UA})/\theta\right)}{(\theta_0/\theta-1)^2 + \mathrm{MSE}(\hat{\theta}_{UA})/\theta^2 - 2(\theta_0/\theta-1)\left(\mathrm{bias}(\hat{\theta}_{UA})/\theta\right)}\,. \tag{5}$$

It may be noted from (5) that $\mathrm{MSE}(\hat{\theta}_{UA})/\theta^2$ and $\mathrm{bias}(\hat{\theta}_{UA})/\theta$ are independent of θ but α_{UA} still depends on θ due to the term θ_0/θ, which can be estimated by $\theta_0/\hat{\theta}_{UA}$ in (5) giving its estimated value $\hat{\alpha}_{UA}$. Substituting $\hat{\alpha}_{UA}$ in place of α_{UA}, we get the shrinkage estimator based on $\hat{\theta}_{UA}$ when a point guess is given with unspecified confidence. This may be written as,

$$\hat{\theta}_{UA(\hat{\alpha})} = \hat{\alpha}_{UA}\theta_0 + (1-\hat{\alpha}_{UA})\hat{\theta}_{UA}\,. \tag{6}$$

Similarly, the shrinkage estimator based on $\hat{\theta}_{BL}$ may be obtained from (6), after replacing $\hat{\theta}_{UA}$ and $\hat{\alpha}_{UA}$ by $\hat{\theta}_{BL}$ and $\hat{\alpha}_{BL}$, respectively, where $\hat{\theta}_{BL}$ is given in (4) and $\hat{\alpha}_{BL}$ can be obtained from (5) after replacing θ and $\hat{\theta}_{UA}$ by $\hat{\theta}_{BL}$. It may be noted here that the expressions for the MSEs of $\hat{\theta}_{UA(\hat{\alpha})}$ and $\hat{\theta}_{BL(\hat{\alpha})}$ can not be obtained and, therefore, one has no option except to go for a simulation study to compare their MSEs.

3 Illustrative Example

For illustration we take the example from Balasubramanian and Balakrishnan (1992), where $n = 30$ items were placed on a life-test experiment and their failure times (in hours) were recorded. The data reported is

0.961	0.990	1.565	2.031	2.204	2.340	3.642	6.008	6.538	7.145
-	-	-	11.937	15.433	18.234	18.307	22.096	-	-
-	28.799	30.692	30.737	33.702	34.245	-	-	-	-

This is a simulated data set from model (3) with $\theta = 20$, where some middle observations were not recorded and the experiment is supposed to be terminated as soon as the 26th item failed. Based on this multiply-II censored sample the estimates are calculated and given in Tables 1 to 3. We see that $\hat{\theta}_{BL(\alpha)}$ or $\hat{\theta}_{UA(\alpha)}$ only provide improvements compared to $\hat{\theta}_{BL}$ or $\hat{\theta}_{UA}$, if the guess θ_0 equals the true value, i.e. $\theta_0 = 20$. For guesses less than 20, the estimates move away from the true value as α increases. Since $\hat{\theta}_{BL}$ and $\hat{\theta}_{UA}$ both underestimate the true parameter, an improvement can be seen if θ_0 is larger than the true value. On the other hand, if no point guess is available and we use $\hat{\theta}_{UA(\hat{\alpha})}$ and $\hat{\theta}_{BL(\hat{\alpha})}$, the estimates only improve if θ_0 is quite close the truth. For other values, the estimates move away from the true value but the magnitude of deviation is smaller as compared to the case when θ_0 is given with specified confidence. However, it may be remarked here that we should not infer about the performance of the estimator on the basis of a single sample. To study the performance of all the estimators we should study the behavior of their MSEs in order to draw fair conclusions.

Table 1: Estimates $\hat{\theta}_{BL(\alpha)}$ based on $\hat{\theta}_{BL}$ when the guess θ_0 is given with confidence α

θ_0	$\hat{\theta}_{BL}$	$\hat{\theta}_{BL(\alpha)}$			
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
16	19.971	19.176	18.382	17.588	16.794
17	19.971	19.376	18.782	18.188	17.594
18	19.971	19.576	19.182	18.788	18.394
19	19.971	19.776	19.582	19.388	19.194
20	19.971	19.976	19.982	19.988	19.994
21	19.971	20.176	20.382	20.588	20.794
22	19.971	20.376	20.782	21.188	21.594
23	19.971	20.576	21.182	21.788	22.394
24	19.971	20.776	21.582	22.388	23.194

Table 2: Estimates $\hat{\theta}_{UA(\alpha)}$ based on $\hat{\theta}_{UA}$ when the guess θ_0 is given with confidence α

θ_0	$\hat{\theta}_{UA}$	$\hat{\theta}_{UA(\alpha)}$			
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
16	19.320	18.656	17.992	17.328	16.664
17	19.320	18.856	18.392	17.928	17.464
18	19.320	19.056	18.792	18.528	18.264
19	19.320	19.256	19.192	19.128	19.064
20	19.320	19.456	19.592	19.728	19.864
21	19.320	19.656	19.992	20.328	20.664
22	19.320	19.856	20.392	20.928	21.464
23	19.320	20.056	20.792	21.528	22.264
24	19.320	20.256	21.192	22.128	23.064

Table 3: Shrinkage estimates when a guess θ_0 is given with unspecified confidence

θ_0	$\hat{\theta}_{BL}$	$\hat{\theta}_{UA}$	$\hat{\theta}_{BL(\hat{\alpha})}$	$\hat{\theta}_{UA(\hat{\alpha})}$
16.0000	19.9712	19.3196	18.0086	17.4298
17.0000	19.9712	19.3196	18.083	17.5565
18.0000	19.9712	19.3196	18.3978	18.0863
19.0000	19.9712	19.3196	19.0564	18.9977
20.0000	19.9712	19.3196	20.0000	19.9602
21.0000	19.9712	19.3196	20.9341	20.654
22.0000	19.9712	19.3196	21.5723	21.0278
23.0000	19.9712	19.3196	21.8686	21.1799
24.0000	19.9712	19.3196	21.9305	21.2098

4 Comparing the MSEs of the Estimators

4.1 Specified Confidence

We now compare the performance of the shrinkage estimators $\hat{\theta}_{UA(\alpha)}$ and $\hat{\theta}_{BL(\alpha)}$ with that of the corresponding approximate maximum likelihood estimators when a point guess is available with specified confidence. Notice that $\mathrm{MSE}(\hat{\theta}_{UA(\alpha)})$ and $\mathrm{MSE}(\hat{\theta}_{BL(\alpha)})$ are both functions of θ, θ_0, n, α, and r_i, $i = 1, \ldots, k$. The MSEs of these estimators have been calculated for various values of θ, n, α, and r_i. A number of values have been assigned to θ_0 so that the relative variation $\phi = (\theta - \theta_0)/\theta$ takes values in $(-0.60(0.20)0.60)$. This was done to provide a wide variation in the values of θ_0 around the truth.

It is noted that as sample size n increases the MSE of the estimators $\hat{\theta}_{UA(\alpha)}$ and $\hat{\theta}_{BL(\alpha)}$ decreases generally, provided the sampling fraction and the type of sample observations do not change too much. It was further noted that as θ increases the MSE increase without affecting the relative performances of the estimators. Therefore, only for $n = 10$ and $\theta = 5$ the MSEs of the estimators have been shown here in Figure 1.

Moreover, if ϕ is close to zero, i.e. if θ_0 is close to θ, the shrinkage estimator $\hat{\theta}_{BL(\alpha)}$ has smaller MSE than $\hat{\theta}_{BL}$ for all choices of α. However, a greater reduction is obtained for large values of α. It may be further noted that for moderate values of ϕ, i.e. for $|\phi| \leq 0.5$, $\hat{\theta}_{BL(\alpha)}$ has always smaller MSE than $\hat{\theta}_{BL}$ for all α. But if $|\phi| \geq 0.5$, the MSE of $\hat{\theta}_{BL(\alpha)}$ may be larger than that of $\hat{\theta}_{BL}$ for large values of α. The range of ϕ for which $\hat{\theta}_{BL(\alpha)}$ has smaller MSE than $\hat{\theta}_{BL}$ can be increased by taking α small, though the magnitude of reduction in MSE also decreases. It is also interesting to note that if the sample contains higher order observations, the greater reduction in MSE is seen for positive values of ϕ, i.e., when θ_0 is smaller than the true value. The situation is reversed when the observed sample contains lower order values. A similar trend can be observed for the MSE of $\hat{\theta}_{UA(\alpha)}$, which is generally smaller than that of $\hat{\theta}_{BL(\alpha)}$ for small values of α. As α increases, $\mathrm{MSE}(\hat{\theta}_{UA(\alpha)})$ remains smaller than $\mathrm{MSE}(\hat{\theta}_{BL(\alpha)})$ for small values of ϕ, but for large value of ϕ the trend is reversed. For $\alpha = 0.9$ both shrinkage estimators have approximately equal MSE. Except for large values of ϕ, $\mathrm{MSE}(\hat{\theta}_{UA(\alpha)})$ is smaller than $\mathrm{MSE}(\hat{\theta}_{BL(\alpha)})$.

4.2 Unspecified Confidence

As already mentioned, although the shrinkage estimators are obtained in closed forms, analytically closed form expressions for their MSE are not available. Therefore, a comparison of their MSEs will be based on results of a simulation study. For this purpose, a Monte Carlo study of 1000 samples each of size 10 was conducted for various values of θ, ϕ, n, k and r_i. The parameter values considered here are the same as in Section 4.1. Notice that a change in the r_i's, for k fixed, results in a change of the magnitude of the MSE. In general, for a fixed number of observations (i.e., k fixed), if the higher order observations are taken, the MSE decreases slightly for almost all estimators (see Figure 2). The amount of decrease, however, differs from estimator to estimator. Further, on the basis of a thorough study of the results, it was noted that the MSEs of all the proposed estimators increase as θ increases but the trend remains more or less the same.

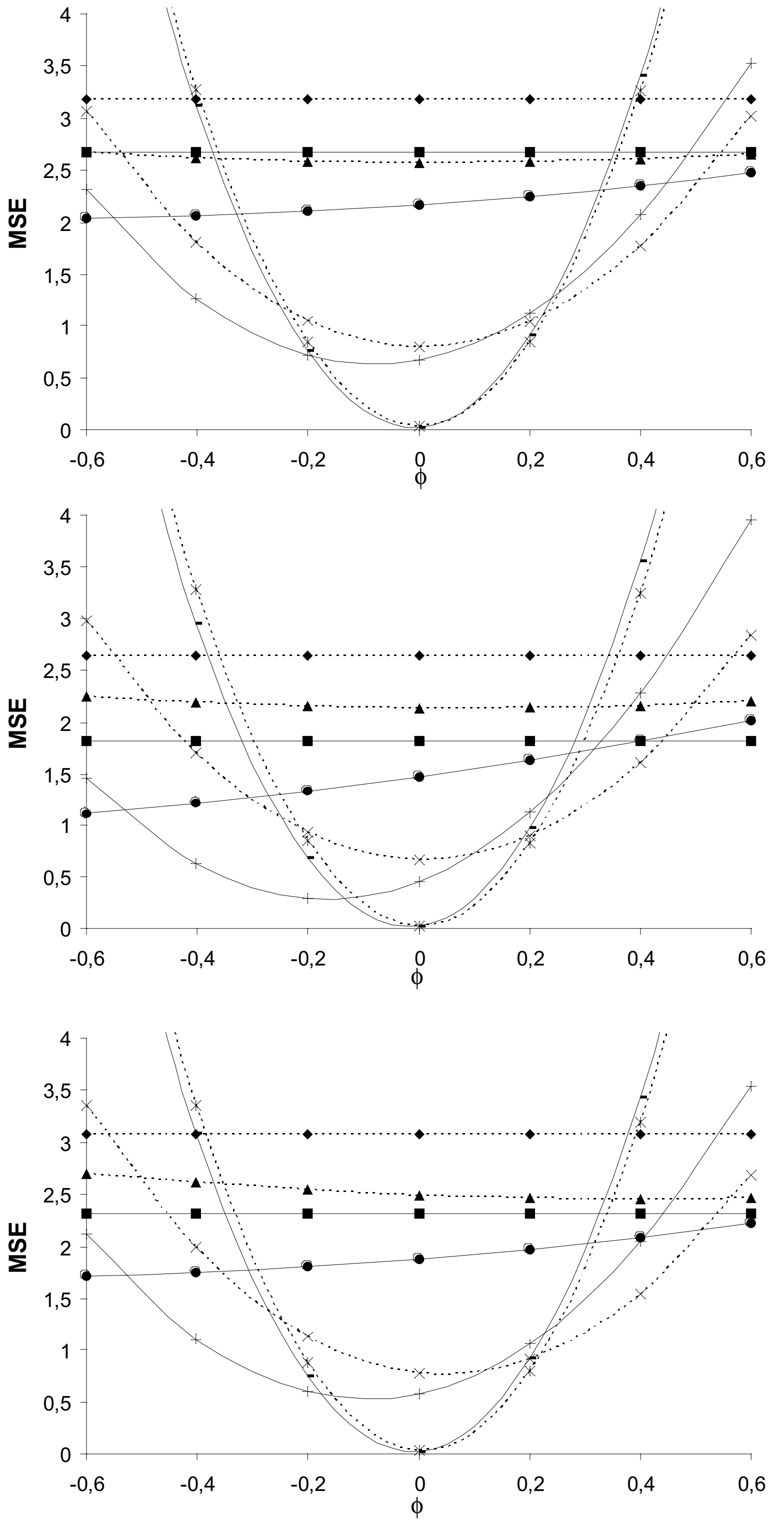

Figure 1: MSEs of the estimators $\hat{\theta}_{BL}$ ($\diamond$), $\hat{\theta}_{BL(\alpha=0.1)}$ ($\triangle$), $\hat{\theta}_{BL(\alpha=0.5)}$ ($\times$), $\hat{\theta}_{BL(\alpha=0.9)}$ ($*$), $\hat{\theta}_{UA}$ ($\square$), $\hat{\theta}_{UA(\alpha=0.1)}$ ($\bullet$), $\hat{\theta}_{UA(\alpha=0.5)}$ ($+$), $\hat{\theta}_{UA(\alpha=0.9)}$ ($-$), for $n = 10$, $\theta = 5$, and $r_i = 1, 2, 5, 6, 8$ (above), $r_i = 1, 2, 6, 9, 10$ (middle), and $r_i = 3, 4, 7, 8, 9$ (below), $i = 1, \ldots, 5$.

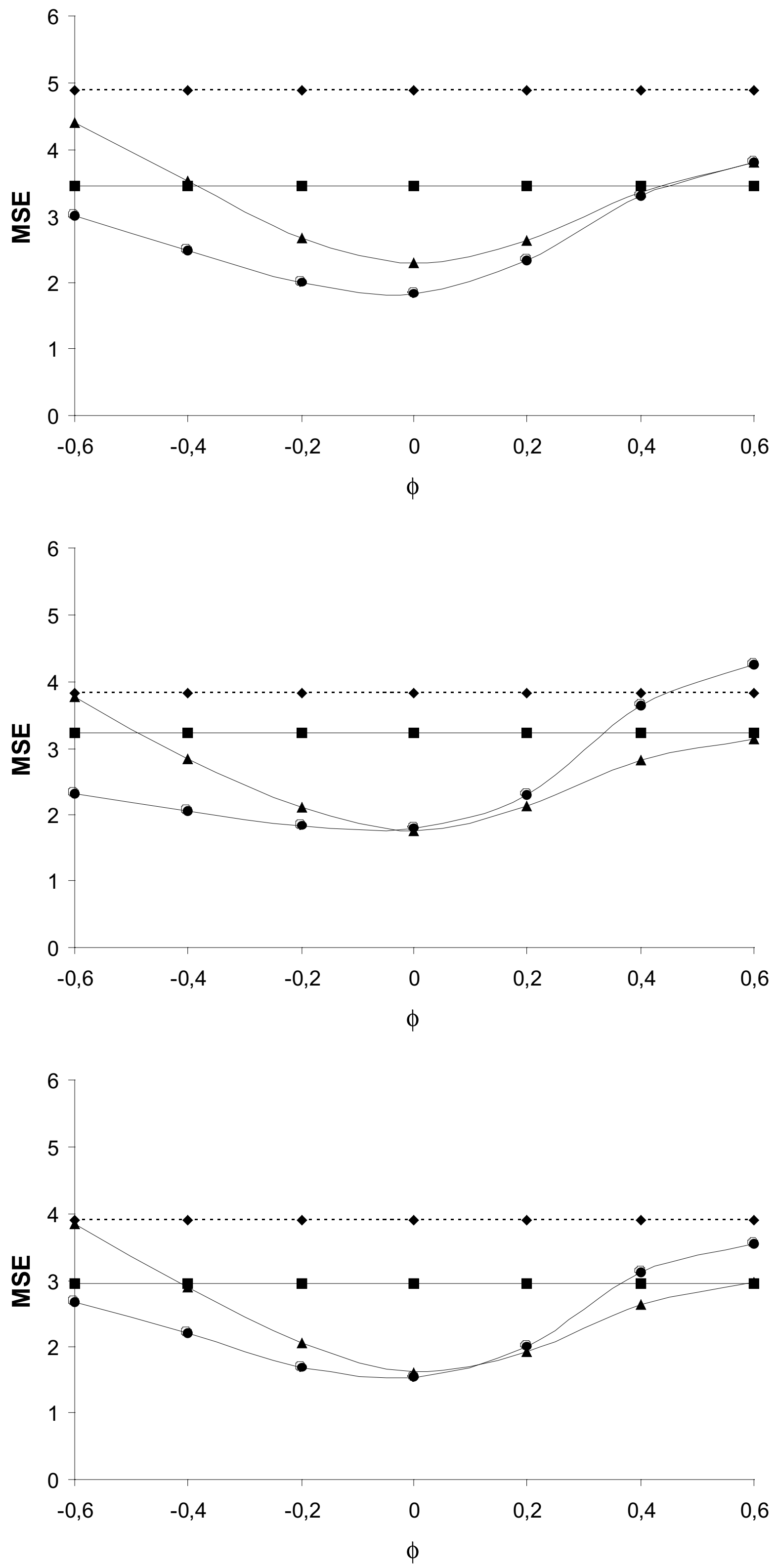

Figure 2: MSEs of the estimators $\hat{\theta}_{BL}$ (◇), $\hat{\theta}_{BL(\alpha)}$ (△), $\hat{\theta}_{UA}$ (□), $\hat{\theta}_{UA(\alpha)}$ (•), for $n = 10$, $\theta = 5$, and $r_i = 1, 2, 5, 6, 8$ (above), $r_i = 1, 2, 6, 9, 10$ (middle), and $r_i = 3, 4, 7, 8, 9$ (below), $i = 1, \ldots, 5$.

As shown in Figure 2, if $\phi = 0$ then $\hat{\theta}_{UA(\hat{\alpha})}$ has the smallest MSE. The MSE of $\hat{\theta}_{BL(\hat{\alpha})}$ is also smaller than the ones of $\hat{\theta}_{UA}$ and $\hat{\theta}_{BL}$. As ϕ increases the MSEs of $\hat{\theta}_{BL(\hat{\alpha})}$ and $\hat{\theta}_{UA(\hat{\alpha})}$ also increase and become larger than those of $\hat{\theta}_{UA}$ and $\hat{\theta}_{BL}$ beyond certain limits of ϕ, say, $\phi \in (\phi_1, \phi_2)$ with $\phi_1 < -0.6$ and $\phi_2 \approx 0.4$. Thus, the shrinkage estimator provides an improvement only in a subspace around the true parameter value. Generally, $\mathrm{MSE}(\hat{\theta}_{UA(\hat{\alpha})})$ is also smaller than $\mathrm{MSE}(\hat{\theta}_{BL(\hat{\alpha})})$ in this subspace. For values of ϕ outside this range, the MSE of $\hat{\theta}_{UA}$ is smaller than the MSEs of all other estimates.

5 Conclusion

From the above results we may conclude that if a prior point guess is close to the truth, we can safely use the shrinkage estimator $\hat{\theta}_{UA(\alpha)}$ together with a large value of α, because it provides the smallest MSE. On the other hand, if the point guess is expected to be in the immediate neighborhood, one can still use $\hat{\theta}_{UA(\alpha)}$. However, if it is suspected that the true value of θ may be far away from the guessed value θ_0, one should never use a shrinkage estimator. In such situations the best one can do is to use $\hat{\theta}_{UA}$, i.e., the approximate maximum likelihood estimator proposed by Singh et al. (2004).

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