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Some Estimators of the Dispersion Parameter of a Chi-distributed Radial Error with Applications to Target Analysis

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Abstract: The dispersion parameter of a chi-distributed radial error is of interest in numerous target analysis problems as a measure of weapon-system accuracy, and it is often of practical importance to estimate it. This paper presents a few classical estimators including the maximum likelihood estimator, an unbiased estimator and a minimum mean squared error estimator of this dispersion for both when the origin or "center of impact" is known or can be assumed as known and when it is unknown. Some families of shrinkage estimators have also been suggested when a prior point estimate of the dispersion parameter is available in addition to sample information. The estimators of circular error probable and spherical error probable have been obtained as well. A simulation study has been carried out to demonstrate the performance of the proposed estimators.

Zusammenfassung: Der Dispersionsparameter eines chi-verteilten radialen Fehlers ist bei Zielanalysen als Maß für die Genauigkeit eines Waffensystems von Interesse. Daher ist es häufig von praktischer Relevanz, diesen Parameter zu schätzen. Wir präsentieren klassische Schätzer wie den Maximum-Likelihood Schätzer, einen unverzerrten Schätzer, und den minimalen mittleren quadratischen Fehler Schätzer für diese Dispersion. Die Schätzer werden für die Situation betrachtet wenn der Nullpunkt, das Einschusszentrum, bekannt ist oder dies angenommen wird und wenn er unbekannt ist. Familien von Shrinkage-Schätzern werden auch vorgeschlagen, falls zusätzlich zur Stichprobeninformation noch eine vorweg Information über die Dispersion verfügbar ist. Wir erhalten Schätzungen für den kreisförmigen und den kugelfgörmigen mutmaßlichen Fehler. Eine Simulationsstudie wird durchgeführt um die Güte der vorgeschlagenen Schätzer zu demonstrieren.

Keywords: Circular Error Probable (CEP), Spherical Error Probable (SEP), Prior Information, Bias, Mean Squared Error.

1 Introduction

The following two paragraphs are taken from Rizos (1999): The uncertainty in a position can be expressed as the probability that the error will not exceed a certain amount.

Under the assumption that position errors follow a normal distribution, this probability can be related to the magnitude of the standard deviation. For example, in the case of a linear (one-dimensional) accuracy measure, one standard deviation or one-sigma would correspond to a 68.27% confidence interval. That is, it is assumed that the mean of an infinitely large sample of position results is the correct result, and the standard deviation of this sample defines the interval on either side of the mean quantity that contains 68.27% of all the results. 31.73% of the results will therefore be outside this range, and if the one-sigma quantity is taken as a measure of accuracy, then 68.27% of the results will be deemed acceptable and the remainder will be outside the accuracy 'specification'. The probabilities of the result being in the interval two-sigma and three-sigma are respectively 95.45% and 99.73% on either side of the mean.

Vertical uncertainty can be expressed in this one-dimensional form. This concept can be extended to two dimensions, so that areas can be constructed corresponding to distinct error probabilities such as 50%, 95%, etc. These zones are centered at the correct or true position. In general these zones are elliptical in shape, and they are known as 'error ellipses' or 'error ellipsoids' depending upon the number of dimensions, see Harvey (1994) for details. However, traditional navigation users have expressed horizontal position uncertainties in the form of circles and three dimensional position uncertainties as spheres. This simplification of the error distribution requires the definition of the radial error or distance root mean square error, which can be determined as

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2}\,, \tag{1}$$

where the components x_k, $k = 1, \ldots, p$, are independently normally distributed about a zero mean and all of them reflect the same variance σ^2. Navigation system errors generally follow a known error distribution. It is well known that $(r/\sigma)^2 \sim \chi_p^2$, i.e.,

$$f_p(r^2/\sigma^2) = \frac{2^{-p/2}}{\Gamma(p/2)} \left(\frac{r}{\sigma}\right)^{2((p/2)-1)} \exp\left\{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right\} .$$

It follows that $(r/\sigma) \sim \chi$ distribution, and for the probability density function (pdf) of r, which is of primary interest here, we have

$$f_p(r) = \frac{2^{-(p-2)/2}}{\sigma\Gamma(p/2)} \left(\frac{r}{\sigma}\right)^{p-1} \exp\left\{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right\} , \qquad 0 \le r \le \infty\,. \tag{2}$$

With reference to $f_p(r)$, σ is to be regarded as a parameter of scale or of dispersion. It is to be noted that the pdf $f_p(r)$ is the generalization of the circular (two-dimensional) and the spherical (three-dimensional) normal variables. Their probability density functions are respectively given by

$$\begin{aligned} f_2(r) &= \left(\frac{r}{\sigma^2}\right) \exp\left\{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right\} \\ f_3(r) &= \sqrt{\frac{2}{\pi}} \left(\frac{r^2}{\sigma^3}\right) \exp\left\{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right\} . \end{aligned} \tag{3}$$

A variety of error measures are used in positioning deriving from different positioning requirements. People are generally most familiar with error measures for a scalar random variable. The term Sigma is equivalent to the estimated standard deviation of a variable. The probability rating varies according to the number of parameters implicit in the variable's pdf. For example, horizontal positioning involves two parameters- latitude and longitude. Vertical positioning relies on only the height component of the position, and therefore only involves one parameter. In navigation and positioning, two-dimensional distributions are of interest for horizontal positioning. Three-dimensional errors are also important, although very often the vertical direction has very different performance requirements and is specified separately.

Consider the problem of directing a projectile, such as a missile, at a target. It obviously is of considerable interest to those involved in strategic planning and targeting to have a measure of the expected accuracy of the projectile. A well known measure of accuracy is the Circular Error Probability (CEP), which is the number r such that on average half of a group of projectiles will fall within the circle of radius r about the target point, see Eckler (1988). When missiles are aimed at a target, the deviations along two orthogonal directions of the impact point from the target center are often assumed to be distributed according to a bivariate normal distribution. Harter (1960), Lowe (1960) and Beyer (1966) gave tables of CEP. Additional information on CEP is provided by Groenewoud et al. (1967).

CEP refers to latitude and longitude (horizontal) position accuracy. A CEP of $1m$ means that the average horizontal position error is $1m$. Another way of stating this is the horizontal position error is less than $1m$ 50% of the time. Thus CEP refers to the radius of a circle in which 50% of the values occur, i.e. if a CEP of $1m$ is quoted then 50% of absolute horizontal point positions should be within $1m$ of the true position. Thus, the CEP is the root of the equation $F_2(r) = 1/2$ and is therefore given by

$$C = \sigma\sqrt{\frac{2\log 2}{\log e}} = \sigma\sqrt{\frac{2 \cdot 0.301029995}{0.434294481}} = 1.1774\sigma\,, \tag{4}$$

where $F_2(r) = 1 - \exp\{-(r/\sigma)^2/2\}$ is the cumulative distribution function (cdf) of the two dimensional radial error $r = (x_1^2 + x_2^2)^{1/2}$.

Scott (1997) notes that "Often it is useful to characterize navigation accuracy in terms of spherical errors. In satellite navigation systems such as GPS and GLONASS, horizontal accuracy is usually much better than vertical accuracy because vertical information is developed from satellites at high elevation angles. Typically, the highest elevation satellite is only at 45 to 50 degree elevation whereas there are a multitude of satellites at lower elevation angles." Else, consider the problem of determining an estimate, which may reflect light on the accuracy of a weapon system directed against an attacking aircraft. Spherical Error Probability (SEP) is the three-dimensional analogue of the probable error of a single variate. Just as the probable error measures the half-width of the mean-centered interval which includes 50 percent of the normal probability mass, the SEP measures the radius of the mean-centered sphere which includes 50 percent of a trivariate normal probability mass. SEP refers to latitude, longitude, and height (3D) position accuracy. This three-dimensional accuracy is commonly referred to as 'bomb on target' since it provides an altitude enhancement over the two-dimensional CEP. Singh (1962) was apparently the first who used term SEP.

The cdf of the three-dimensional radial error $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ is

$$F_3(r) = \int_0^r f_3(u)\, du = \sqrt{\frac{2}{\pi}} \int_0^r \frac{u^2}{\sigma^3} \exp\left\{ -\frac{1}{2}\left(\frac{u}{\sigma}\right)^2 \right\} du \,, \tag{5}$$

where $f_3(u)$ is given in (3). As discussed in Singh (1970), the transformation $(u/\sigma)^2/2 = t$, ($u = \sqrt{2t}\sigma$, $du = \sigma/\sqrt{2t}\, dt$), gives

$$F_3(r) = \frac{1}{\Gamma(3/2)} \int_0^{r^2/2\sigma^2} t^{(3/2)-1} e^{-t}\, dt = I\left(\frac{r^2}{2\sigma^2}, \frac{3}{2}\right) ,$$

where $I(\cdot,\cdot)$ is the incomplete gamma function.

Further substituting $u/\sigma = x$, ($u = x\sigma$, $du = \sigma\, dx$), in (5) results in

$$F_3(r) = \frac{2}{\sqrt{2\pi}} \int_0^{r/\sigma} x^2 e^{-x^2/2}\, dx = 2\left[G\left(\frac{r}{\sigma}\right) - H_1\left(\frac{r}{\sigma}\right) g\left(\frac{r}{\sigma}\right) \right] ,$$

where $G(r/\sigma) = (2\pi)^{-1/2} \int_0^{r/\sigma} \exp(-x^2/2)\, dx$, $g\,(r/\sigma) = (2\pi)^{-1/2} \exp\left(-r^2/2\sigma^2\right)$, and $H_1(r/\sigma) = r/\sigma$ is the Hermite polynomial of the first order.

The cdf can also be expressed as

$$F_3(r) = 2\left[G\left(\frac{r}{\sigma}\right) + g^{(1)}\left(\frac{r}{\sigma}\right) \right] \tag{6}$$

so that the pdf of r is

$$f_3(r) = \frac{2}{\sigma}\left[g\left(\frac{r}{\sigma}\right) + g^{(2)}\left(\frac{r}{\sigma}\right) \right] ,$$

where $g^{(b)}(\psi) = d^b g(\psi)/d\psi^b$. The value of r/σ for which $F_3(r) = 1/2$ is obtained from (6) with the help of the tables of standard Gaussian function and found to be 1.5382, as noted in Singh (1970). Thus the spherical error probability (SEP) is defined as

$$S = 1.5382\sigma \,. \tag{7}$$

Suppose that the sample is from a population with distribution according to (2). Thus, observed values of r are unrestricted so that any non-negative value can be measured. Cohen (1955) reported the maximum likelihood estimator (MLE) of σ, defined as

$$\hat{\sigma}_{mle} = \sqrt{\frac{1}{np} \sum_{i=1}^{n} r_i^2} \,, \tag{8}$$

where r_i^2, as defined in (1), is the i-th observed squared miss distance.

The relative bias (RB), relative variance (RV) and relative mean squared error (RMSE) of an estimator t of the parameter σ are defined as

$$\begin{aligned} \mathrm{RB}(t) &= \frac{\mathrm{bias}(t)}{\sigma} = \frac{\mathrm{E}(t) - \sigma}{\sigma} \\ \mathrm{RV}(t) &= \frac{\mathrm{var}(t)}{\sigma^2} = \frac{\mathrm{E}(t - \mathrm{E}(t))^2}{\sigma^2} \\ \mathrm{RMSE}(t) &= \frac{\mathrm{MSE}(t)}{\sigma^2} = \frac{\mathrm{E}(t-\sigma)^2}{\sigma^2} \,. \end{aligned}$$

Further, the absolute relative bias (ARB) is

$$\mathrm{ARB}(t) = \left|\frac{\mathrm{bias}(t)}{\sigma}\right| = \left|\frac{\mathrm{E}(t)-\sigma}{\sigma}\right| .$$

The formula for the r-th moment about zero (see, e.g., Kotz and Johnson, 1982, p. 439) for the χ_p-distributed random variable X is given by

$$\mathrm{E}\left(X^r\right) = 2^{r/2}\frac{\Gamma[(p+r)/2]}{\Gamma(p/2)} . \tag{9}$$

Since $np\hat{\sigma}^2_{mle}/\sigma^2 \sim \chi^2_{np}$, we get $(np)^{1/2}\hat{\sigma}_{mle}/\sigma \sim \chi_{np}$. Now using (9) we have

$$\mathrm{E}\left(\hat{\sigma}^j_{mle}\right) = \sigma^j\left(\frac{2}{np}\right)^{j/2} K_{(n,p,j)} , \tag{10}$$

where $K_{(n,p,j)} = \Gamma\left[(np+j)/2\right]/\Gamma(np/2)$ and $j \neq 0$ is a real number.

Thus, the RB, RV, and RMSE of $\hat{\sigma}_{mle}$ are given by

$$\mathrm{RB}(\hat{\sigma}_{mle}) = \sqrt{\frac{2}{np}}K_{(n,p,1)} - 1 \tag{11}$$

$$\mathrm{RV}(\hat{\sigma}_{mle}) = 1 - \frac{2}{np}K^2_{(n,p,1)} \tag{12}$$

$$\mathrm{RMSE}(\hat{\sigma}_{mle}) = 2\left(1 - \sqrt{\frac{2}{np}}K_{(n,p,1)}\right) . \tag{13}$$

It is clear from the results (4) and (7) that the estimation of CEP and SEP is essentially the estimation of the dispersion parameter of the radial error in two and three dimensions, respectively. Substitution of $p = 2$ and $p = 3$ in (8) and using (4) and (7) yield the MLE

$$\hat{C}_{mle} = 1.1774\sqrt{\frac{1}{2n}\sum_{i=1}^{n}(x^2_{1i}+x^2_{2i})} \tag{14}$$

$$\hat{S}_{mle} = 1.5382\sqrt{\frac{1}{3n}\sum_{i=1}^{n}(x^2_{1i}+x^2_{2i}+x^2_{3i})} \tag{15}$$

It is trivial to obtain the RB, RV, and RMSE of $\hat{C}_{mle}$ and $\hat{S}_{mle}$ by putting $p = 2$ and $p = 3$ in (11), (12) and (13).

This paper deals with the problem of estimating the dispersion parameter of a chi-distributed radial error when the population mean is known and also when it is unknown. Some classical estimators along with a class of shrinkage type estimators are proposed with their characteristics in both the cases. The estimators of CEP and SEP are derived.

2 Estimators Based on Sample Information when the Center of Impact is Known

Many authors together with Chapman and Robbins (1951) and Cohen Jr. (1955) defined conventional estimators including the uniformly minimum variance unbiased estimator (UMVUE), the minimum mean squared error (MMSE) estimator, etc., in a class of invariant estimators, viz.

$$\hat{\sigma} = c\sqrt{\sum_{i=1}^{n} r_i^2}\,, \tag{16}$$

with c being a constant. We define a class of estimators of σ as

$$\hat{\sigma}_A = A\hat{\sigma}_{mle}\,, \tag{17}$$

where A is a constant to be determined such that $\mathrm{MSE}\,(\hat{\sigma}_A)$ is minimizedt. It is to be noted that for $A = (np/2)^{1/2}K_{(n,p,1)}$, (17) reduces to the UMVUE in the class (16) as

$$\hat{\sigma}_{unb} = \frac{1}{K_{(n,p,1)}}\sqrt{\frac{1}{2}\sum_{i=1}^{n} r_i^2} \tag{18}$$

with

$$\mathrm{RV}\,(\hat{\sigma}_{unb}) = \frac{np}{2}\frac{1}{K^2_{(n,p,1)}} - 1 = \mathrm{RMSE}\,(\hat{\sigma}_{unb})\;.$$

The mean squared error

$$\mathrm{MSE}\,(\hat{\sigma}_A) = \sigma^2\left(A^2 - 2A\sqrt{\frac{2}{np}}K_{(n,p,1)} + 1\right)$$

is minimized for $A = (2/(np))^{1/2}K_{(n,p,1)}$. Thus, the resulting MMSE estimator in the class (17) is

$$\hat{\sigma}_{mms} = \sqrt{\frac{2}{np}}K_{(n,p,1)}\sqrt{\frac{1}{np}\sum_{i=1}^{n} r_i^2}\,, \tag{19}$$

with

$$\mathrm{RB}\,(\hat{\sigma}_{mms}) = \frac{2}{np}K^2_{(n,p,1)} - 1\,, \qquad \mathrm{RMSE}\,(\hat{\sigma}_{mms}) = 1 - \frac{2}{np}K^2_{(n,p,1)}\,. \tag{20}$$

Substituting $p = 2$ and $p = 3$ in (18) and with (4) and (7), gives unbiased estimators

$$\hat{C}_{unb} = \frac{1.1774}{K_{(n,2,1)}}\sqrt{\frac{1}{2}\sum_{i=1}^{n}(x_{1i}^2 + x_{2i}^2)} \tag{21}$$

$$\hat{S}_{unb} = \frac{1.5382}{K_{(n,3,1)}}\sqrt{\frac{1}{2}\sum_{i=1}^{n}(x_{1i}^2 + x_{2i}^2 + x_{3i}^2)}\,, \tag{22}$$

which are due to Chapman and Robbins (1951) and Moranda (1959).

Similarly, by virtue of (4), (7), and (19) the MMSE estimators of CEP and SEP are

$$\hat{C}_{mms} = 1.1774 \frac{K_{(n,2,1)}}{n} \sqrt{\frac{1}{2} \sum_{i=1}^{n} (x_{1i}^2 + x_{2i}^2)} \tag{23}$$

$$\hat{S}_{mms} = 1.5382 \frac{2K_{(n,3,1)}}{3n} \sqrt{\frac{1}{2} \sum_{i=1}^{n} (x_{1i}^2 + x_{2i}^2 + x_{3i}^2)}\,, \tag{24}$$

which are due to Singh (1992) and Singh and Upadhyaya (2003). Respective RV's, RB's, and RMSE's can be determined as before.

So far we have discussed the classical estimation procedures that were aimed towards the use of sample information alone. An attempt may be made to combine the sample information with other relevant aspects of the problem in order to obtain better estimates. The prior information of the parameter may provide us with the estimators better than the classical estimators, if used intelligibly. In certain circumstances, this information may prove to be an invaluable asset in improving the efficiency of the estimators manifold.

3 Estimators Based on Prior Information when the Center of Impact is Known

It is well known that the defence and space research organizations keep very up-to-date and systematic data about the operations related with targeting a missile or a bomb or positioning of satellites, etc. To test the efficiency of a defence weapon against an air attack or the efficiency of an aircraft attacking a target on the land or in the sea, observations are generally recorded in the form of deviations of the impact point from the target center. Due to this considerable handling of positioning data, one may have a reliable estimate of the dispersion parameter σ. Generally, this guessed or specified value of σ, say σ_0, comes from past experiments about similar situations involving similar parameter.

We define the following class of estimators, viz.

$$\hat{\sigma}_{(\alpha,\beta)} = \sigma_0 \left[\alpha + W \left(\frac{\hat{\sigma}_{unb}}{\sigma_0} \right)^{\beta} \right]\,, \tag{25}$$

where α and β are real numbers such that $0 < \alpha < \infty$ and $\beta \neq 0$, W is a constant to be determined such that

$$\begin{aligned} \mathrm{MSE}\left(\hat{\sigma}_{(\alpha,\beta)}\right) &= \mathrm{E}\left(-\sigma(1-\lambda\alpha) + W\sigma_0^{1-\beta}\hat{\sigma}_{unb}^{\beta}\right)^2 \\ &= \sigma^2(1-\lambda\alpha)^2 + W^2\sigma_0^{2(1-\beta)}\mathrm{E}(\hat{\sigma}_{unb}^{2\beta}) - 2W(1-\lambda\alpha)\sigma\sigma_0^{1-\beta}\mathrm{E}(\hat{\sigma}_{unb}^{\beta})\,, \end{aligned} \tag{26}$$

with $\lambda = \sigma_0/\sigma$, is at its minimum. Using (10) we get

$$\mathrm{MSE}\left(\hat{\sigma}_{(\alpha,\beta)}\right) = \sigma^2 \left[(1-\lambda\alpha)^2 + W^2\lambda^{2(1-\beta)} \frac{K_{(n,p,2\beta)}}{K_{(n,p,1)}^{2\beta}} - 2W(1-\lambda\alpha)\lambda^{(1-\beta)} \frac{K_{(n,p,\beta)}}{K_{(n,p,1)}^{\beta}} \right]\,,$$

which is minimized for

$$W = (1-\alpha\lambda)\lambda^{\beta-1}W_{(n,p,\beta)}\ , \tag{27}$$

where $W_{(n,p,\beta)} = K^{\beta}_{(n,p,1)}K_{(n,p,\beta)}/K_{(n,p,2\beta)}$. As W in (27) is a function of the unknown parameter σ, therefore replacing σ by its unbiased estimator $\hat{\sigma}_{unb}$, we get an estimate of W which now becomes random, as

$$\hat{W} = \left\{1-\alpha\frac{\sigma_0}{\hat{\sigma}_{unb}}\right\}\left(\frac{\sigma_0}{\hat{\sigma}_{unb}}\right)^{\beta-1}W_{(n,p,\beta)}\ .$$

Substitution of $\hat{W}$ in place of W in (25) yields a workable form of the estimator as

$$\begin{aligned}\hat{\hat{\sigma}}_{(\alpha,\beta)} &= \alpha\sigma_0 + W_{(n,p,\beta)}\left(\hat{\sigma}_{unb}-\alpha\sigma_0\right)\\ &= \alpha\sigma_0\left(1-W_{(n,p,\beta)}\right)+\hat{\sigma}_{unb}W_{(n,p,\beta)}\ ,\end{aligned} \tag{28}$$

with

$$\mathrm{ARB}\left(\hat{\hat{\sigma}}_{(\alpha,\beta)}\right) = \left|(\alpha\lambda-1)(1-W_{(n,p,\beta)})\right| \tag{29}$$

$$\mathrm{RMSE}\left(\hat{\hat{\sigma}}_{(\alpha,\beta)}\right) = (\alpha\lambda-1)^2\left(1-W_{(n,p,\beta)}\right)^2+\left\{\frac{np}{2K^2_{(n,p,1)}}-1\right\}W^2_{(n,p,\beta)}\ . \tag{30}$$

Thus, by virtue of (4), (7), (21), (22), and (28), we define the estimators

$$\hat{\hat{C}}_{(\alpha,\beta)} = 1.1774\alpha\sigma_0\left(1-W_{(n,2,\beta)}\right)+\hat{C}_{unb}W_{(n,2,\beta)} \tag{31}$$

$$\hat{\hat{S}}_{(\alpha,\beta)} = 1.5382\alpha\sigma_0\left(1-W_{(n,3,\beta)}\right)+\hat{S}_{unb}W_{(n,3,\beta)}\ . \tag{32}$$

Putting $p=2$ and $p=3$ in (29) and (30), the ARB's and RMSE's of $\hat{\hat{C}}_{(\alpha,\beta)}$ and $\hat{\hat{S}}_{(\alpha,\beta)}$ can easily be obtained.

Many estimators can be generated from (28) by substituting different values of (α,β). The following points are of interest in this regard:

1. It can be easily proved that $0 \le W_{(n,p,\beta)} \le 1$, implying that the suggested estimator is the convex combination of $\alpha\sigma_0$ and $\hat{\sigma}_{unb}$.

2. The smaller the values of $W_{(n,p,\beta)}$ the less biased and more efficient the estimators are. However, $W_{(n,p,\beta)}$ is an increasing function in n, i.e., $W_{(n,p,\beta)} \to 1$ as $n \to \infty$.

3. The proposed estimator boils down to the unbiased estimator for $W_{(n,p,\beta)} = 1$.

4. Thus, β may be chosen such that $0 < W_{(n,p,\beta)} < 1$, $W_{(n,p,\beta)} \to 0$, and $\beta > -np/2$.

5. The suggested estimator is not only unbiased but renders maximum gain in efficiency if $\alpha = \lambda^{-1}$. Thus, $\alpha = 1$ represents very strong belief in the guessed value illustrating that $\sigma = \sigma_0$. Hence, it is recommended to select α close to 1. However, if $\sigma \ll \sigma_0$, then $\alpha \ll 1$ would give better results, whereas if $\sigma \gg \sigma_0$ then $\alpha \gg 1$ would give certainly good results.

6. The RMSE of the suggested class of estimators (28), or in particular those of (31) and (32), increases as sample size increases. In other words, the relative efficiency of these classes of estimators decreases with increasing sample size. Thus, the proposed classes of estimators give better results for smaller samples.

3.1 Special Case

For $(\alpha, \beta) = (1, 1)$, the class (28) reduces to

$$\hat{\hat{\sigma}}_{(1,1)} = \sigma_0 + W_{(n,p,1)} \left(\hat{\sigma}_{unb} - \sigma_0\right) ,$$

with $W_{(n,p,1)} = 2(np)^{-1} K^2_{(n,p,1)}$. Putting $(\alpha, \beta) = (1, 1)$ in (30) gives

$$\mathrm{RMSE}\left(\hat{\hat{\sigma}}_{(1,1)}\right) = \left(1 - \frac{2}{np} K^2_{(n,p,1)}\right) \left[(1-\lambda)^2 \left(1 - \frac{2}{np} K^2_{(n,p,1)}\right) + \frac{2}{np} K^2_{(n,p,1)}\right] .$$

With (20) it follows that $\hat{\hat{\sigma}}_{(1,1)}$ is more efficient than $\hat{\sigma}_{mms}$ in terms of the RMSE, if $(1-\lambda)^2 \left(1 - 2K^2_{(n,p,1)}/np\right) + 2K^2_{(n,p,1)}/np \le 1$, i.e., if $0 < \lambda \le 2$, i.e., if

$$\frac{\sigma_0}{2} \le \sigma < \infty ,$$

which shows that the proposed estimator $\hat{\hat{\sigma}}_{(1,1)}$ is better than the MMSE estimator $\hat{\sigma}_{mms}$ for a wider range of σ, i.e. for $\sigma \in (\sigma_0/2, \infty)$.

4 Estimators Based on Sample Information when the Center of Impact is Unknown

In case the population mean is not equal to zero, the estimators discussed in Section 1 and 2 would give biased results. Therefore, deviations should be computed from the mean of the sample so as to minimize the bias. If $x_1, \ldots, x_p$ follow a p-variate normal law with unknown mean (or aim point) $(\mu_1, \ldots, \mu_p)$ and unknown variance $\sigma_1^2 = \cdots = \sigma_p^2 = \sigma^2$, the MLE of σ is given by

$$\tilde{\sigma}_{mle} = \sqrt{\frac{1}{np} \sum_{i=1}^{n} \ddot{r}_i^2} .$$

where $\ddot{r}_i^2 = \sum_{j=1}^{p} (x_{ji} - \bar{x}_j)^2$. Since $np\tilde{\sigma}^2_{mle}/\sigma^2 \sim \chi^2_{p(n-1)}$, the variable $(np)^{1/2} \tilde{\sigma}_{mle}/\sigma \sim \chi_{p(n-1)}$.

Now, we define a class of estimators of σ as

$$\tilde{\sigma}_B = B\tilde{\sigma}_{mle} , \tag{33}$$

where B is a constant to be chosen such that $\mathrm{MSE}(\tilde{\sigma}_B)$ is at its minimum. It is to be noted that for $B = (np/2)^{1/2}/K_{(m,p,1)}$, (33) reduces to the UMVUE in a class of invariant estimators $\ddot{\sigma} = k \left(\sum_{i=1}^{n} \ddot{r}_i^2\right)^{1/2}$, k being a constant, as

$$\tilde{\sigma}_{unb} = \frac{1}{K_{(m,p,1)}} \sqrt{\frac{1}{2} \sum_{i=1}^{n} \ddot{r}_i^2} ,$$

where $K_{(m,p,1)} = \Gamma[(mp+1)/2]/\Gamma(mp/2)$ with $m = (n-1)$. Since

$$\mathrm{MSE}\left(\tilde{\sigma}_B\right) = \sigma^2 \left[\frac{m}{n}B^2 - 2B\sqrt{\frac{2}{mn}}K_{(m,p,1)} + 1\right] ,$$

the optimal choice of B is

$$B = \frac{n}{m}\sqrt{\frac{2}{mp}}K_{(m,p,1)} .$$

The MMSE estimator in the class (33) is therefore

$$\tilde{\sigma}_{mms} = \sqrt{\frac{2}{mp}}K_{(m,p,1)}\sqrt{\frac{1}{mp}\sum_{i=1}^{n}\ddot{r}_i^2} .$$

If we replace r_i^2 in (14) and (15) by $\ddot{r}_i^2$, $i = 1, \ldots, n$, $p = 2$ or 3, we get MLE's of CEP and SEP (see, Singh, 1970). Similarly, if we replace n by m and r_i^2 by $\ddot{r}_i^2$ in (21), (22), (23), and (24) we get unbiased estimators (see, Chapman and Robbins, 1951, Moranda, 1959, or Singh, 1970) and MMSE estimators of CEP and SEP (see, Singh, 1992, and Singh and Upadhyaya, 2003).

The RB's, RV's, and RMSE's of the estimators discussed in this section can easily be obtained by simply replacing n by m in the respective expressions obtained for the case when the center of impact is known.

5 Estimators Based on Prior Information when the Center of Impact is unknown

If the center of impact is unknown, we define the following class of estimators

$$\tilde{\sigma}_{(u,v)} = \sigma_0\left[u + H\left(\frac{\tilde{\sigma}_{unb}}{\sigma_0}\right)^v\right] ,$$

where u and v are real numbers such that $0 < u < \infty$ and $v \neq 0$, H is a constant to be determined such that $\mathrm{MSE}\left(\tilde{\sigma}_{(u,v)}\right)$ is at its minimum. Proceeding similar as in Section 3 yields the class of shrinkage estimators

$$\tilde{\tilde{\sigma}}_{(u,v)} = u\sigma_0\left(1 - H_{(m,p,v)}\right) + \tilde{\sigma}_{unb}H_{(m,p,v)} ,$$

where $H_{(m,p,v)} = K^v_{(m,p,1)}K_{(m,p,v)}/K_{(m,p,2v)}$ and with

$$\mathrm{ARB}\left(\tilde{\tilde{\sigma}}_{(u,v)}\right) = \left|(u\lambda - 1)\left(1 - H_{(m,p,v)}\right)\right| \tag{34}$$

$$\mathrm{RMSE}\left(\tilde{\tilde{\sigma}}_{(u,v)}\right) = (u\lambda - 1)^2\left(1 - H_{(m,p,v)}\right)^2 + \left(\frac{mp}{2K^2_{(m,p,1)}} - 1\right)H^2_{(m,p,v)} . \tag{35}$$

Similarly, estimators of CEP and SEP are

$$\tilde{\tilde{C}}_{(u,v)} = 1.1774u\sigma_0\left(1 - H_{(m,2,v)}\right) + \tilde{C}_{unb}H_{(m,2,v)}$$
$$\tilde{\tilde{S}}_{(u,v)} = 1.5382u\sigma_0\left(1 - H_{(m,3,v)}\right) + \tilde{S}_{unb}H_{(m,3,v)} .$$

Table 1: Simulated sample from N(0, 4) (left), and from N(2, 9) (right).

x_1	x_2	x_3	x_1	x_2	x_3
-0.12132	0.07292	-0.88389	-2.11850	4.60404	-2.44222
-2.59037	0.70944	0.13006	-2.07938	5.09405	6.68754
0.98285	-1.96912	-3.27462	2.50136	1.08946	-0.77591
-0.66351	-1.59366	0.11381	3.01007	2.32918	-2.70473
2.15499	-1.62687	-2.42294	-3.39691	-0.32858	4.31897
3.63641	3.24631	-2.02326	5.79597	-0.00442	-0.08140
-1.39969	1.87469	-2.29120	-2.25021	3.61843	0.01476
1.97335	-3.08520	-1.57044	0.91700	-0.88697	3.87766
-3.00996	1.09687	-1.03358	1.71327	0.26837	1.73793
-1.94124	0.05899	0.22130	2.22638	0.90818	3.63013
-2.47672	-0.91997	1.64590	7.40968	1.07960	4.22995
-4.03386	2.95965	-0.62516	2.57249	-3.02871	-5.04124
-1.75506	1.53138	4.00505	1.29061	2.91704	1.90670
2.18448	-2.16321	2.18615	3.52185	-1.14687	2.54703
-0.14279	-0.05761	2.31286	1.53862	1.43802	-0.90783

Table 2: Estimates of the SEP with true value 3.0764 and their characteristics when center of impact is known (left), and with true value 4.6146 when center is unknown (right).

	Estimate	ARB	RMSE	Estimate	ARB	RMSE
$\hat{S}_{mle}$	3.123414	0.005740	0.011479	4.669214	0.005734	0.011469
$\hat{S}_{unb}$	3.141445	0.000000	0.011579	4.696169	0.000000	0.011568
$\hat{S}_{mms}$	3.105486	0.011446	0.011446	4.642413	0.011435	0.011436
$\hat{S}_{(1,-2)}$	3.130325	0.001958	0.009840	4.721853	0.007064	0.009740
$\hat{S}_{(1,-1)}$	3.138219	0.000568	0.011059	4.703558	0.002033	0.011015
$\hat{S}_{(1,1)}$	3.139820	0.000286	0.011316	4.699633	0.000953	0.011306
$\hat{S}_{(1,2)}$	3.133849	0.001338	0.010375	4.713248	0.004698	0.010323

Their RB and RMSE is obtained from (34) and (35).

Various estimators can be generated from $\tilde{\tilde{\sigma}}_{(u,v)}$ by considering different values for (u, v). A discussion of results on $\tilde{\tilde{\sigma}}_{(u,v)}$ follows from Section 3.

6 An Example

There is no other better choice than a simulated sample to test the performance of the proposed estimators, as real target analysis data are not available owing to confidentiality norms. In this section a particular case of $p = 3$ has been considered and thereby estimation of SEP has been performed. Some of the estimators of the proposed classes, viz., $\hat{S}_{(\alpha,\beta)}$ and $\tilde{S}_{(u,v)}$ have been generated and it is found that these estimators have emerged as an improvement over both, the MLE and the MMSE estimators.

Considering the case when center of impact is known, a set of three random samples of 15 observations each is generated from a normal population with mean 0 and variance 4. Assuming that the prior point estimate of the dispersion parameter is given as $\sigma_0 = 1.95$, the findings are summarized in the left part of Table 2. We further considered the case when center of impact is nonzero and unknown. A set of three random samples of 15 observations each is generated from a normal population with mean 2 and variance 9. Assuming that the prior point estimate of the dispersion parameter is given as $\sigma_0 = 3.25$, the findings are in the right part of Table 2.

It is evident from the results displayed in Table 2 that the proposed MMSE estimators are better than the previously presented estimators of Chapman and Robbins (1951), Cohen Jr. (1955), Moranda (1959), Singh (1992), and Singh and Upadhyaya (2003). If reliable prior information is available, the proposed shrinkage type estimators are less biased and more efficient than the conventional estimators near the guessed value and possibly worse farther away.

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